Luis Felipe Tabera Alonso
(ed.)

Discrete Mathematics Days 2022

Ediciones Universidad Cantabria

Discrete Mathematics Days 2022

Colección Difunde # 263

Discrete Mathematics Days 2022

Luis Felipe Tabera Alonso

(editor)

Ediciones
Universidad
Cantabria

Discrete Mathematics Days (2022 : Santander), autor.

Discrete Mathematics Days 2022 / Luis Felipe Tabera Alonso (editor). – Santander : Editorial de la Universidad de Cantabria, 2022

323 páginas. – (Difunde ; 263)

Incluye los abstracts extensos de los trabajos presentados a los Discrete Mathematics Days, encuentro celebrado en Santander del 4 al 6 de julio de 2022.

ISBN 978-84-19024-03-9

1. Matemática discreta - Congresos. I. Tabera, Luis Felipe, editor de compilación.

519.6(063)

THEMA: PBD, 1DSE-ES-F, 3MRBH

Digitization: Manuel Ángel Ortiz Velasco [emeaov]

© Editorial de la Universidad de Cantabria
Avda. de los Castros, 52 - 39005 Santander. Cantabria (España)
www.editorial.unican.es
ISNI: 0000 0005 0686 0180

ISBN: 978-84-19024-03-9 (PDF)

ISBN: 978-84-19024-02-2 (DGB)

DOI: https://doi.org/10.22429/Euc2022.016

Hecho en España-*Made in Spain*

Santander, 2022

Contents

Foreword **1**

Committees **3**

1 Plenary Talks **5**

Contributed Talks **9**

2 Extending a conjecture of Graham and Lovász on the distance characteristic polynomial **11**
Aida Abiad, Boris Brimkov, Sakander Hayat, Antonina P. Khramova, Jack H. Koolen

3 Unit disk visibility graphs **17**
Deniz Ağaoğlu Çağırıcı, Onur Çağırıcı

4 Polynomials for marked graphs and the chromatic symmetric function **24**
José Aliste-Prieto, Anna de Mier, Rosa Orellana, José Zamora

5 An analogue of Chvátal's Hamiltonicity theorem for randomly perturbed graphs **31**
Alexander Allin, Alberto Espuny Díaz

6 On sumsets of nonbases of maximum size **37**
Béla Bajnok, Péter Pál Pach

7 On the Heesch number in $\mathbb{E}^d$ **42**
Bojan Bašić, Anna Slivková

8 2-LC triangulated manifolds are exponentially many **46**
Bruno Benedetti, Marta Pavelka

9 Accessible set systems and a conjecture on Cohen-Macaulay binomial edge ideals **49**
Davide Bolognini, Antonio Macchia, Giancarlo Rinaldo, Francesco Strazzanti

10 Logarithmic convergence of projective planes **54**
Márton Borbényi, Panna Fekete, Aranka Hrušková, Ander Lamaison

11 Seymour's second neighborhood conjecture in arbitrary orientations of a random graph **58**
Fábio Botler, Phablo F. S. Moura, Tássio Naia

12 Competitive equilibrium always exists for combinatorial auctions with graphical pricing schemes **64**
Marie-Charlotte Brandenburg, Christian Haase, Ngoc Mai Tran

13 Conway's fried potato problem: a (quadratic) algorithm leading to an optimal division for convex polygons **71**
Antonio Cañete, Isabel Fernández, Alberto Márquez

14 Enumeration of chordal planar graphs and maps **77**
Jordi Castellví, Marc Noy, Clément Requilé

15 Characterizing the extremal families in Erdős–Ko–Rado theorems **84**
Gilad Chase, Neta Dafni, Yuval Filmus, Nathan Lindzey

16 Combinatorics and preservation of conically stable polynomials (Extended abstract) **90**
Giulia Codenotti, Stephan Gardoll, Thorsten Theobald

17 Generalised flatness constants: a framework applied in dimension 2 **97**
Giulia Codenotti, Thomas Hall, Johannes Hofscheier

18 Multitriangulations and tropical Pfaffians **104**
Luis Crespo Ruiz, Francisco Santos

19 A parallel between the descriptive complexities of finite groups and Latin square graphs **110**
Anuj Dawar, Danny Vagnozzi

20 On vertex bisection width of random d-regular graphs **116**
Josep Díaz, Öznur Yaşar Diner, Maria Serna, Oriol Serra

21 Sidon-Ramsey & B_h-Ramsey numbers **123**
Manuel A. Espinosa-García, Amanda Montejano, Edgardo Roldán-Pensado, J. David Suárez

22 Powers of Hamilton cycles in graphs perturbed by a random geometric graph **128**
Alberto Espuny Díaz, Joseph Hyde

23 Maximum running times for graph bootstrap percolation processes **134**
David Fabian, Patrick Morris, Tibor Szabó

24 Homomorphisms between graphs embedded in surfaces (extended abstract) **141**
Delia Garijo, Andrew Goodall, Lluís Vena

25 How to build a pillar: a proof of Thomassen's conjecture **147**
Irene Gil Fernández, Hong Liu

26 Rainbow solutions of a linear equation with coefficients in $\mathbb{Z}/p\mathbb{Z}$ **153**
Mario Huicochea

27 The weighted sum method for multi-objective optimization using Test Sets via Gröbner Bases **159**
José Manuel Jiménez-Cobano, Haydee Jiménez-Tafur, José María Ucha-Enríquez

28 Generalized Permutahedra and Positive Flag Dressians **163**
Michael Joswig, Georg Loho, Dante Luber, Jorge Olarte

29 Common systems of two equations over the binary field **169**
Daniel Král', Ander Lamaison, Péter Pál Pach

30 Linking+SensoGraph: A new graph-based method for sensory analysis **174**
Jacob Lahne, David Orden, Katherine Phetxumphou, Marino Tejedor-Romero

31 Random projections for the distance geometry problem **181**
Leo Liberti, Benedetto Manca, Pierre-Louis Poirion

32 On discrete Borell-Brascamp-Lieb type inequalities for big negative parameters **188**
Eduardo Lucas Marín

33 New results for the spectra of weighted graphs of order 5 **193**
Carlos Marijuán, Miriam Pisonero

34 Local parking procedures on the integers **200**
Philippe Nadeau

35 Extremal Sidon sets are Fourier uniform, with applications to partition regularity **206**
Miquel Ortega, Sean Prendiville

36 Many regular triangulations and many polytopes **211**
Arnau Padrol, Eva Philippe, Francisco Santos

37 Deformation cones of graphical zonotopes **217**
Arnau Padrol, Vincent Pilaud, Germain Poullot

38 New Ramsey multiplicity bounds and search heuristics **224**
Olaf Parczyk, Sebastian Pokutta, Christoph Spiegel, Tibor Szabó

39 Fast positive Plücker trees **231**
Julian Pfeifle

40 Generalized Perron roots and solvability of the absolute value equation **237**
Manuel Radons, Josué Tonelli-Cueto

41 Normal limiting distributions for systems of linear equations in random sets **243**
Juanjo Rué, Maximilian Wötzel

42 On a nonabelian Kneser theorem **249**
Oriol Serra, Maximilian Wötzel

43 A faster algorithm for the two cluster partitioning problem **255**
Francisco-Javier Soto, Ana I. Gómez, Domingo Gómez-Pérez

Poster Presentations **261**

44 Cospectral generalized Johnson and Grassmann graphs **263**
Aida Abiad, Jozefien D'haeseleer, Robin Simoens

45 Advances on the 3-color off-diagonal generalized Schur numbers $S(3; k_1, k_2, k_3)$ **269**
T. Ahmed, L. Boza, M. P. Revuelta, M. I. Sanz

46 On bipartite biregular Moore graphs **274**
Gabriela Araujo-Pardo, Cristina Dalfó, Miguel Àngel Fiol, Nacho López

47 Computation of 2D discrete geometric moments through inclusion-exclusion **280**
Lidija Čomić, Paola Magillo

48 Trees having domination number equal to $\{K_2\}$-isolation number **286**
Adriana Dapena, Magdalena Lemańska, María José Souto-Salorio, Francisco Vazquez-Araujo

49 Restrained differential of a graph **291**
Magda Dettlaff, Abel Cabrera-Martínez, Magdalena Lemańska, Juan Alberto Rodríguez-Velázquez

50 Distance-equalizer sets of graphs **295**
A. González, C. Hernando, M. Mora

51 The discrete side of Distance Geometry: a focus on the 1-dimensional case **301**
Simon B. Hengeveld, A. Mucherino

52 On maximum independent sets of almost bipartite graphs **306**
Vadim E. Levit, Eugen Mandrescu

53 On local bipartite Moore graphs **313**
Nacho López, Josep Conde, Gabriela Araujo-Pardo

54 Sequences related to square and cube zig-zag shapes **318**
László Németh, László Szalay

Foreword

This is the book of extended abstracts of the plenary talks and the accepted contributions of the *Discrete Math Days*, which will take place in Santander in July 4–6, 2022.

It has been a long way since we started preparing this conference three years ago, initially intended to take place in June 2020. It was a difficult decision to cancel them in March 2020, only a couple of weeks before our submission period closed and when already a couple of participants had paid their registration fees.

Our initial announcement said that the new dates would "most probably be in June 2021". Alas, it turned out that prediction was too optimistic. An online edition would have been possible in June 2021 but we (the organizing committee and the program committee) preferred to wait a bit more and have a full on-site edition as initially planned. We certainly appreciate the efforts of many colleagues in different parts of the world who organized online conferences in 2021 and gave the community a sense of normality and continuity but, after all, we believe that the most important part of a scientific conference is the personal interaction among participants. This is particularly true for the younger participants, who still need to get to know and be known by their community.

We are very happy with the outcome. The 53 nice contributions in these proceedings, selected by the program committee from a total of 62 submissions, testify to the good health of discrete mathematics and the the high regard that the community has for this meeting. We thank the organizing and program committees for their good work in difficult circumstances, and the four plenary speakers for their understanding and availability when it came to cancelling the old dates and fixing the new ones.

We also want to thank the institutions that have contributed financially to make this possible: our university, faculty and department, the municipality of Santander and its convention center, and the Spanish National Research Agency (MCIN/ AEI /10.13039/501100011033) who supports the conference via grants PID2019-110633GB-I00 and PID2019-106188GB-I00.

Santander, June 1st 2022

Luis Felipe Tabera, Editor of the proceedings
Francisco Santos, Chair of the organizing committee

Committees

Program committee

Chair: Marc Noy, Universitat Politècnica de Catalunya

Albert Atserias , Universitat Politècnica de Catalunya.
Christine Bachoc, Université de Bordeaux.
Sergio Cabello, Univerza v Ljubljani.
Anna de Mier, Universitat Politècnica de Catalunya.
Shalom Eliahou, Université du Littoral Côte d'Opale.
Delia Garijo, Universidad de Sevilla.
Bernardo González Merino, Universidad de Murcia.
Dan Král', Masaryk University Brno.
Carlos Marijuán, Universidad de Valladolid.
Juan Alberto Rodríguez Velázquez, Universitat Rovira i Virgili.
Francisco Santos, Universidad de Cantabria.
Maya Stein, Universidad de Chile.
Anusch Taraz, Technische Universität Hamburg.
Thorsten Theobald, Goethe-Universität Frankfurt.
Lluis Vena Cros, Universiteit van Amsterdam.
Julia Wolf, University of Cambridge.

Organizing committee

Chair: Francisco Santos Leal, Universidad de Cantabria.

Luis Crespo Ruiz, Universidad de Cantabria.
Domingo Gómez Pérez, Universidad de Cantabria.
Ana Isabel Gómez Pérez, Universidad Rey Juan Carlos.
Luis Felipe Tabera Alonso, Universidad de Cantabria.

Plenary Talks
One graph to rule them all: forbidden structures and universal graphs

Marthe Bonamy

Université de Bordeaux

Abstract

Consider all planar graphs on n vertices. What is the smallest graph that contains them all as induced subgraphs?

In this talk, we will gently introduce the audience to the notion of so-called universal graphs (graphs containing all graphs of a given family as induced subgraphs), and focus on the case of graph classes defined by forbidden structures. We present positive and negative results both in dense graphs and in sparse graphs. The audience will also have the answer to the question at the beginning of this abstract, recently established in a breakthrough paper of Dujmović, Esperet, Joret, Gavoille, Micek and Morin

Wandering on random digraphs

János Pach

EPFL Lausanne and Rényi Institute Budapest

Abstract

Random walks on random graphs is an active area of research with numerous results in the last 20 years. While the theory is well developed for undirected graphs, the directed case is much less understood. Unfortunately, networks that arise in applications are often directed by nature (e.g. World Wide Web, citation network, ...). In recent years, new techniques have been developed to study the directed setting. In this talk I will survey some of the most prominent results in the area, including the study of the random stationary measure and the mixing properties of the chain. Time permitting, I will cover two topics more in-depth. First, I will show how directed edges dramatically increase the time required to visit every vertex of the graph. Second, I will present a mathematical framework for the so-called *'power-law hypothesis'*, a property relating in-degrees with Pagerank – the webpage ranking at the core of Google's search engine.

The results presented are joint work with Xing Shi Cai, Pietro Caputo and Matteo Quattropani.

Crossing lemmas for multigraphs?

Guillem Perarnau

Universitat Politècnica de Catalunya

Abstract

The celebrated Crossing Lemma of Thurston and Ajtai, Chvátal, Newborn, and Szemerédi gives an asymptotically tight lower bound on the number of edge crossing in any drawing of a graph G in the plane which has n vertices and $e > 3n - 6$ edges. The lemma has found many applications in topological graph theory, additive number theory, combinatorial geometry, and elsewhere. In order to extend its range of applications, for more than 40 years, there have been many attempts to strengthen this lemma for various special classes of graphs, and to generalize it to multigraphs (where there can be several edges between any pair of vertices). We describe some recent efforts in this direction, and raise several open problems.

Symmetry in discrete structures: graphs, groups and algorithms

Pascal Schweitzer

TU Darmstadt

Abstract

The concept of symmetry is ubiquitous in the study of discrete combinatorial objects. The ensemble of symmetries of an object G is its automorphism group $\mathrm{Aut}(G)$. It provides us with structural information of the object. The group also captures how its symmetries relate to one another. While symmetries are interesting to study in their own right, symmetries also have direct applications in diverse areas that make use of combinatorial objects.

In the talk, I will describe various facets of research surrounding the structure, detection and application of symmetries.

- Regarding the *structure of symmetry*, I will discuss automorphism groups of graphs in minor-closed graph classes. Examples of such graph classes include the class of trees, planar graphs, graphs of bounded genus (i.e., graphs embeddable into a fixed surface without crossings), and graphs of bounded tree width. While a classic result of Frucht shows that every finite group is the automorphism group of a finite graph, Babai showed that with a fixed graph class that excludes a fixed minor not all groups can be represented as automorphism groups. The talk will discuss general structure theorems for the automorphism group of minor-closed graph classes.

- Regarding *detection of symmetry*, it is well known that the so-called graph isomorphism problem, which asks us to decide whether two given graphs are isomorphic, is universal in the following sense. The computation of the automorphism group of an arbitrary (explicitly given) combinatorial object reduces, in polynomial time, to the graph isomorphism problem. The isomorphism problem in turn reduces to a related problem, the problem of computing a canonical labeling. In some contexts this is referred to as computing a normal form of a given graph. The talk will discuss some recent advances in the research on the theory and implementation of practical graph isomorphism solvers.

- Regarding *application of symmetry*, I will focus on a particular type of application in finite model theory. Specifically, I will touch on the quest for a logic capturing Ptime and discuss how it relates to the graph isomorphism and graph canonization problems.

This talk is in part an overview over various results obtained within the ERC-Project EngageS (`https://www.mathematik.tu-darmstadt.de/EngageS`) and based on joint work with numerous co-authors.

Contributed Talks

Extending a conjecture of Graham and Lovász on the distance characteristic polynomial

Aida Abiad[1*,†,‡], Boris Brimkov[2§], Sakander Hayat[3¶], Antonina P. Khramova[4*], Jack H. Koolen[5‖]

[*]Department of Mathematics and Computer Science, Eindhoven University of Technology, The Netherlands
[†]Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Belgium
[‡]Department of Mathematics and Data Science, Vrije Universiteit Brussel, Belgium
[§]Department of Mathematics and Statistics, Slippery Rock University, U.S.A.
[¶]Faculty of Engineering Sciences, GIK Institute of Engineering Sciences and Technology, Topi, Swabi, 23460, Pakistan
[‖]School of Mathematical Sciences, University of Science and Technology of China, Hefei, China

The full version of this work will be published elsewhere.

Abstract

Graham and Lovász conjectured in 1978 that the sequence of normalized coefficients of the distance characteristic polynomial of a tree of order n is unimodal with the maximum value occurring at $\lfloor \frac{n}{2} \rfloor$. In this paper we extend this old conjecture to block graphs. In particular, we prove the unimodality part and we establish the peak for several extremal cases of block graphs with small diameter.

1 Introduction

In [6], Graham and Lovász have studied the coefficients of the distance characteristic polynomial of a tree T on n vertices. In particular, they proved a previously conjectured [5] formula for the coefficients which depends on the number of certain subtrees of T. The formula for the coefficient of x^k also has a factor $(-1)^n 2^{n-k-2}$ for $k \in \{0, \ldots, n\}$. It thus makes sence to consider normalized coefficients of the distance characteristic polynomial of tree, which are obtained from original coefficients by dividing by $(-1)^n 2^{n-k-2}$ for every k.

Definition 1. A sequence $a_0, a_1, a_2, \ldots, a_n$ of real numbers is **unimodal** if there is a k such that $a_{i-1} \leq a_i$ for $i \leq k$ and $a_i \geq a_{i+1}$ for $i \geq k$. A sequence is **log-concave** if $a_j^2 \geq a_{j-1}a_{j+1}$ for all $j = 1, \ldots, n-1$.

In the same paper [6], Graham and Lovász conjectured that the sequence of normalized coefficients of T is unimodal with the maximum value occurring at $\lfloor \frac{n}{2} \rfloor$ for a tree T of order n. Later, Collins [3] confirmed the conjecture for the star S_n, and also showed that for a path P_n, the sequence is unimodal with a maximum value at $(1 - \frac{1}{\sqrt{5}})n$. Thus, Graham and Lovász conjecture was reformulated as follows.

[1]Email: a.abiad.monge@tue.nl. Research of A. A. partially supported by the FWO (Research Foundation Flanders), grant number 1285921N.

[2]Email: boris.brimkov@sru.edu

[3]Email: sakander.hayat@giki.edu.pk.

[4]Email: a.khramova@tue.nl. Research of A. K. supported by the NWO (Dutch Science Foundation), grant number OCENW.KLEIN.475.

[5]Email: koolen@ustc.edu.cn. Research of J. K. partially supported by the National Natural Science Foundation of China (No. 12071454), Anhui Initiative in Quantum Information Technologies (No. AHY150000) and the National Key R and D Program of China (No. 2020YFA0713100).

Conjecture 2. [6, 3] The coefficients of the distance characteristic polynomial of any tree with n vertices are unimodal with peak between $\lfloor \frac{n}{2} \rfloor$ and $\lceil (1 - \frac{1}{\sqrt{5}})n \rceil$.

Aalipour et al. [1] confirmed the unimodality part of the Graham-Lovász conjecture by proving that the sequence of normalized coefficients of a tree is indeed always unimodal; in fact, they proved the stronger statement that this sequence is log-concave.

A natural generalization of trees are clique trees, or block graphs.

Definition 3. A connected graph is a *block graph* or *clique tree* if its blocks, or 2-connected components, are cliques.

In this paper we investigate Conjecture 2 with regard to block graphs. In particular, we establish the unimodality part of Conjecture 2 for block graphs, extending the corresponding result for trees by Aalipour *et al.* [1]. We also show the peak part of Conjecture 2 holds for several extremal classes of block graphs with small diameter, extending Collins' result for star graphs [3].

2 Preliminaries

Throughout this paper, $G = (V, E)$ will denote an undirected, simple, loopless graph with n vertices. The distance matrix of G, denoted by $D(G)$, is the matrix indexed by the vertices of G whose (i, j)-entry equals the distance between the vertices v_i and v_j. The characteristic polynomial of $D(G)$ is denoted by $p_{D(G)}(x)$, and c_k is the coefficient of x^k in this polynomial.

Definition 4. Given two graphs G and H, their **Cartesian product** is the graph $G \Box H$ whose vertex set is $V(G) \times V(H)$ and whose edges are the pairs $((a, x), (b, y))$ with $a, b \in V(G)$, $x, y \in V(H)$ and either $(a, b) \in E(G)$ and $x = y$, or $a = b$ and $(x, y) \in E(H)$.
The Cartesian product $H_1 \Box \cdots \Box H_k$ of graphs $H_1, \ldots, H_k$ is also denoted $\prod_{h=1}^{k} H_h$.

The Cartesian product of d copies of K_n is also known as the Hamming graph. In particular, a hypercube is the Cartesian product of d copies of K_2.

With any connected graph G a metric space can be associated.

Definition 5. The **path metric** d_G of G is defined for all vertices $u, v \in V$, $d_G(u, v)$ as the distance between u and v in G. Then (V, d_G) is the **graphic metric space** associated with G.

Finally, we will make use of the following well-known result; it has been proved, *e.g.*, in [2, Lemma 1.1] with the additional assumption that the polynomial coefficients are nonnegative, but it is straightforward to remove that assumption.

Lemma 6. **(i)** *If $p(x) = a_n x_n + \cdots + a_1 x + a_0$ is a real-rooted polynomial, then the coefficient sequence a_i of p is log-concave.*

(ii) *If the sequence $a_0, a_1, a_2, \ldots, a_n$ is positive and log-concave, then it is unimodal.*

3 On ℓ_1-embeddability of metric spaces

In this section we develop a general theory regarding metric spaces, particularly the metric hierarchy of spaces whose distance matrix have exactly one positive eigenvalue. The main result of this section, Theorem 11, is later used to show the unimodality of Graham and Lovász conjecture for block graphs.

Definition 7. A metric space (X, d) is **of negative type** if for all weight functions $w : X \to \mathbb{Z}$ with $\sum_{x \in X} w(x) = 0$ we have

$$\sum_{x \in X} \sum_{y \in X} w(x) w(y) d(x, y) \leq 0. \tag{1}$$

If the same inequality holds for all weight functions with $\sum_{x\in X} w(x) = 1$, then we say that (X, d) is **hypermetric**.

It is fairly easy to show that (X, d) being hypermetric implies it is of negative type.

Definition 8. A metric space (X, d) is said to be **ℓ_1-embeddable** if it can be embedded isometrically into the ℓ_1-space $(\mathbb{R}^m, d_{\ell_1})$ for some integer $m \geq 1$. Here, d_{ℓ_1} is defined by

$$d_{\ell_1}(x, y) := \sum_{1\leq i\leq m} |x_i - y_i| \quad \text{for } x, y \in \mathbb{R}^m.$$

Note that ℓ_1-embeddability of (X, d) implies it is hypermetric [9].

Definition 9. A metric space (V_n, d) is **hypercube embeddable** if it can be isometrically embedded into the space $(\{0, 1\}^m, d_{\ell_1})$ for some $m \geq 1$, i.e., if there are n binary vectors $v_1, \ldots, v_n \in \{0, 1\}^m$ such that

$$d(i, j) = d_{\ell_1}(v_i, v_j) \text{ for all } i, j \in V_n.$$

It can be shown that hypercube embeddability implies ℓ_1-embeddability.

We have the following implications:

$$\begin{array}{c} (X, d) \text{ is hypercube embeddable} \\ \Downarrow \\ (X, d) \text{ is } \ell_1\text{-embeddable} \\ \Downarrow \\ (X, d) \text{ is hypermetric} \\ \Downarrow \\ (X, d) \text{ is of negative type} \\ \Downarrow \\ \text{The distance matrix of } G \text{ has exactly one positive eigenvalue} \end{array}$$

Let (X_1, d_1) and (X_2, d_2) be two metric spaces. Their direct product is the metric space $(X_1 \times X_2, d_1 \otimes d_2)$ where, for $x_1, y_1 \in X_1$, $x_2, y_2 \in X_2$,

$$d_1 \otimes d_2((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2). \tag{2}$$

Note that for path metrics, the direct product operation corresponds to the Cartesian product of graphs. Namely, if G and H are two connected graphs, then the direct product of their path metrics coincides with the path metric of the Cartesian product of G and H.

This lemma provides a relation between the metric hierarchy and the direct product of respective metric spaces.

Lemma 10. *Let d_i be a distance on the set X_i, for $i = 1, 2$. Then $(X_1 \times X_2, d_1 \otimes d_2)$ is of negative type if and only if both (X_1, d_1) and (X_2, d_2) are of negative type.*

The proof is technical and follows from definitions of a metric space of negative type and a direct product of metric spaces.

Theorem 11. *Let G be the a graph whose 2-connected components are of negative type. If $D(G)$ is the distance matrix of G, then $D(G)$ has exactly one positive eigenvalue.*

Proof. Lemma 10 ensures us that the Cartesian product of the 2-connected components of G is of negative type. Finally, we just need to observe that G is an isometric subgraph of the Cartesian product, and the result follows immediately. □

We note without proof that both Lemma 10 and Theorem 11 also hold if the 2-connected components are hypercube, ℓ_1-embeddable or hypermetric.

Lin, Liu, and Lu have shown that the distance matrix of a block graph has exactly one positive eigenvalue [8, Theorem 3.2]. Since the 2-connected components of a block graph are ℓ_1-embeddable and are thus of negative type, Theorem 11 generalizes this result.

4 Graham and Lovász conjecture for block graphs

4.1 Unimodality for block graphs

In this section we show that the sequence of coefficients of the distance characteristic polynomial of a block graph is unimodal. We begin with a preliminary result.

Lemma 12. *The coefficients of the distance characteristic polynomial of a block graph G satisfy*

$$(-1)^{n-1}c_k > 0 \quad \textit{for } 0 \le k \le n-2.$$

Proof. From Theorem 11 the distance matrix $D(G)$ of a block graph G has one positive and $n-1$ negative eigenvalues. We now extend the argument given in [5, Theorem 2.3] that $(-1)^{n-1}c_k(T) > 0$ for $0 \le k \le n-2$ for a tree T, given that its distance matrix has one positive and $n-1$ negative eigenvalues. Let the eigenvalues of $D(G)$ for a block graph G be denoted by $\lambda_1, -\lambda_2, \ldots, -\lambda_n$, where $\lambda_i > 0$ for $i = 1, \ldots, n$. Then the distance characteristic polynomial is

$$\begin{aligned}\det(D(G) - xI) &= (-1)^n(x-\lambda_1)(x+\lambda_2)\ldots(x+\lambda_n) = \\ &= (-1)^n(x-\lambda_1)\sum_{k=0}^{n-1} g_{n-1-k}x^k = \\ &= (-1)^n\left(x^n + \sum_{k=1}^{n-1}(g_{n-k} - \lambda_1 g_{n-k-1})x^k - \lambda_1 g_{n-1}\right),\end{aligned}$$

where g_k is the sum of all k-fold products of $\lambda_2, \ldots, \lambda_n$. Then, $c_{n-1}(G) = g_1 - \lambda_1$, but also $c_{n-1} = -c_n \mathrm{tr}(D) = 0$; thus, $g_1 = \lambda_1$. Then, since $g_{n-k} - \lambda_1 g_{n-k-1} = g_{n-k} - g_1 g_{n-k-1} < 0$ for $k = 1, \ldots, n-2$, and since for $k = 0$, $-\lambda_1 g_{n-1} = -g_1 g_{n-1} = -\prod_{i=1}^{n}\lambda_i < 0$, it follows that $(-1)^{n-1}c_k > 0$ for $0 \le k \le n-2$. □

Theorem 13. *For a block graph G, the sequence of coefficients of the distance characteristic polynomial $(-1)^{n-1}c_0, \ldots, (-1)^{n-1}c_{n-2}$ is unimodal.*

Proof. First, it follows from Lemma 12 that if G is a block graph, then the coefficients of the distance characteristic polynomial satisfy $(-1)^{n-1}c_k > 0$ for $0 \le k \le n-2$.

Since the distance matrix D is a real symmetric matrix, the distance characteristic polynomial is real-rooted. From Lemma 6(i), it follows that the sequence of coefficients is log-concave. Moreover, since

$$(-1)^{n-1}c_k > 0 \quad \text{for } 0 \le k \le n-2,$$

then Lemma 6(ii) implies that the sequence of coefficients of the distance characteristic polynomial is unimodal. □

4.2 Peak location for extremal classes of block graphs

In this section we focus on establishing the peak location part of Conjecture 2 for several families of block graphs: the union of two cliques by a vertex, the barbell and lollipop graphs, and graphs of one large clique, two pendant vertices and a single cut vertex.

Theorem 14. *Let G be a graph obtained by adding a universal vertex to the disjoint union of two cliques K_t, and let $n = |V(G)| = 2t + 1$. Then the sequence of coefficients of the distance characteristic polynomial of G is unimodal with peak at $\lfloor \frac{n}{2} \rfloor$.*

Proof. Using the quotient matrix, we can compute the distance characteristic polynomial of G to be

$$\begin{aligned} p_D(x) &= (x+1)^{2t-2}(-(t+1)-x)(-2t-(3t-1)x+x^2) = \\ &= (x+1)^{2t-2}(ax^3+bx^2+cx+d), \end{aligned}$$

where $a = -1$, $b = 2t-2$, $c = 3t^2+4t-1$, and $d = 2t^2+2t$. Multiplying the binomial expansion of $(x+1)^{2t-2}$ by (ax^3+bx^2+cx+d) and combining the coefficients of terms with the same power, we obtain the coefficient of x^{t-j} for $t \geq 4$ and $j \geq 0$,

$$c_{t-j} = a\binom{2t-2}{t-3-j} + b\binom{2t-2}{t-2-j} + c\binom{2t-2}{t-1-j} + d\binom{2t-2}{t-j}. \tag{3}$$

Using the formula $\binom{n}{k} = \frac{(n-k+1)}{k}\binom{n}{k-1}$, we can rewrite (3) as

$$\begin{aligned} c_{t-j} = \binom{2t-2}{t-3-j} \Bigg(a + b\cdot\frac{j+t+1}{t-j-2} + c\cdot\frac{j+t+1}{t-j-2}\cdot\frac{j+t}{t-j-1} + \\ + d\cdot\frac{j+t-1}{t-j-2}\cdot\frac{j+t}{t-j-1}\cdot\frac{j+t+1}{t-j}\Bigg) = \\ = \binom{2t-2}{t-3-j}\cdot f(t,j), \end{aligned}$$

where $f(t,j) = \frac{t(j+t)\left(t\left(j^2+j(3-4t)-t(5t+11)+2\right)+2\right)}{(j-t)(j-t+1)(j-t+2)}$. Then, $c_{t-j} \geq c_{t-(j+1)}$ if and only if

$$\binom{2t-2}{t-3-j}\cdot f(t,j) \geq \binom{2t-2}{t-3-(j+1)}\cdot f(t,j+1),$$

which is equivalent to

$$\frac{f(t,j)}{t-3-j} \geq \frac{f(t,j+1)}{t+j+2}. \tag{4}$$

It can be verified by symbolic algebra for rational functions that (4) holds for all integers j and t with $t \geq 4$ and $0 \leq j < t-3$; for $t \leq 3$, it can be verified that $c_{t-j} \geq c_{t-(j+1)}$ by explicitly computing the distance characteristic polynomial. Similarly as above, it can be shown that $c_{t+j} \geq c_{t+(j+1)}$ for all t and $j \geq 0$. Thus, it follows that $c_0 \leq \ldots \leq c_t \geq \ldots \geq c_{2t+1}$, and hence the distance characteristic polynomial of G is unimodal with peak at $t = \lfloor \frac{2t+1}{2} \rfloor = \lfloor \frac{n}{2} \rfloor$. □

Definition 15. A **barbell graph** $B(t,\ell)$ is obtained by connecting two cliques K_t with a path on ℓ vertices by identifying the leaves of the path with one of the vertices in each clique. A **lollipop graph** $L(t,\ell)$ is obtained by adding a path on ℓ vertices to a single clique K_t so that one of the leaves of the path is also a vertex of the clique.

To locate the peak of the coefficients of the distance characteristic polynomial for a barbell or a lollipop graph, an approach similar to one in Theorem 14 can be used: namely, a quotient partition is considered, and an explicit formula for the coefficient can be obtained using the characteristic polynomial of the quotient matrix and the results on the spectrum of G, see [7, Theorems 3.1, 3.3]. Knowing the formula, we consider inequality $c_i \geq c_{i-1}$ and directly show that it holds if $i = p$ and does not hold if $i = p+1$, where p is the conjectured peak location. Unimodality of the sequence of coefficients then implies that these two inequalities are sufficient to locate the peak. To compare, the methods that are used by Collins to locate the peak for families of trees in [3] are based on structural properties of the distance matrix itself and do not require identifying the distance spectrum and the distance polynomial of a graph.

Theorem 16. *Let G be a barbell graph $B(t,\ell)$, so that $|V(G)| = n = 2t + \ell - 2$. Then the sequence of coefficients of the distance characteristic polynomial of G is unimodal with peak at $t-1$ if $\ell = 2$ and at t if $\ell \in \{3,4,5\}$.*

Corollary 17. *Let G be a lollipop graph $L(t,\ell)$, so that $|V(G)| = n = t + \ell - 1$. Let $\ell \in \{2,3,4,5\}$. Then the sequence of coefficients of the distance characteristic polynomial of G is unimodal with peak at $\lfloor \frac{n-1}{2} \rfloor$.*

Finally, let G be a block graph with a single cut vertex, one t-clique, and m 2-cliques. The next result shows the peak location for the sequence of coefficients in case $m = 2$.

Theorem 18. *Let G be a block graph obtained by adding two edges incident to one vertex of a clique of size t, so that there is exactly one cut vertex in G and the number of vertices is $n = t + 2$. Then the the sequence of coefficients of the distance characteristic polynomial of G is unimodal with peak at $\lceil \frac{n}{2} \rceil$.*

The proof follows a method similar to one demonstrated for the classes of block graphs considered previously, apart from a few technical details which arise from differences in the distance spectrum of the graph.

Based on the results in this section and SageMath simulations, we end up by posing the following question.

Question 19. *Are the coefficients of the distance characteristic polynomial of any block graph with n vertices unimodal with peak between $\lfloor \frac{n}{3} \rfloor$ and $\lceil \frac{n}{2} \rceil$?*

References

[1] G. Aalipour, A. Abiad, Z. Berikkyzy, L. Hogben, F. H. J. Kenter, J. C.-H. Lin, and M. Tait. "Proof of a conjecture of Graham and Lovász concerning unimodality of coefficients of the distance characteristic polynomial of a tree". In: *Electron. J. Linear Algebra* 34 (2018), pp. 373–380. DOI: 10.13001/1081-3810.3493.

[2] P. Brändén. "Unimodality, log-concavity, real-rootedness, and beyond". In: *Handbook of enumerative combinatorics*. Ed. by M. Bóna. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, 2015, pp. xxiii+1061.

[3] K. L. Collins. "On a conjecture of Graham and Lovász about distance matrices". In: *Discrete Appl. Math.* 25.1-2 (1989), pp. 27–35. DOI: 10.1016/0166-218X(89)90044-9.

[4] M. M. Deza and M. Laurent. *Geometry of cuts and metrics*. Vol. 15. Algorithms and Combinatorics. Springer-Verlag, Berlin, 1997, pp. xii+587. DOI: 10.1007/978-3-642-04295-9.

[5] M. Edelberg, M. R. Garey, and R. L. Graham. "On the distance matrix of a tree". In: *Discrete Math.* 14.1 (1976), pp. 23–39. DOI: 10.1016/0012-365X(76)90003-0.

[6] R. L. Graham and L. Lovász. "Distance matrix polynomials of trees". In: *Adv. in Math.* 29.1 (1978), pp. 60–88. DOI: 10.1016/0001-8708(78)90005-1.

[7] L. Hogben and C. Reinhart. *Spectra of variants of distance matrices of graphs and digraphs: a survey*. Preprint. 2021. DOI: 10.48550/ARXIV.2103.00647.

[8] H. Lin, R. Liu, and X. Lu. "The inertia and energy of the distance matrix of a connected graph". In: *Linear Algebra Appl.* 467 (2015), pp. 29–39. DOI: 10.1016/j.laa.2014.10.045.

[9] P. Terwilliger and M. Deza. "The classification of finite connected hypermetric spaces". In: *Graphs Combin.* 3.3 (1987), pp. 293–298. DOI: 10.1007/BF01788552.

Unit disk visibility graphs

Deniz Ağaoğlu Çağırıcı[1*], Onur Çağırıcı[2†]

[*]Masaryk University, Brno, Czech Republic
[†]Toronto Metropolitan University (formerly, Ryerson University), Toronto, Canada

This is an extension of the paper [1] published in EUROCOMB21.

Abstract

We study unit disk visibility graphs, where the visibility relation between a pair of geometric entities is defined by not only obstacles, but also the distance between them. This particular graph class models real world scenarios more accurately compared to the conventional visibility graphs. We first define and classify the unit disk visibility graphs, and then show that the 3-coloring problem is NP-complete when unit disk visibility model is used for a set of line segments (which applies to a set of points) even when every segment is axis-parallel (i.e., parallel to either x- or y-axis) or when every segment has unit length (i.e., 1-unit long), and for a polygon with holes.

1 Introduction

A visibility graph is a simple graph $G = (V, E)$ defined over a set $\mathcal{P} = \{p_1, \ldots, p_n\}$ of n geometric entities where a vertex $u \in V$ represents a geometric entity $p_u \in \mathcal{P}$, and the edge $uv \in E$ exists if and only if p_u and p_v are mutually visible (or see each other). In the literature, visibility graphs were studied considering various geometric sets such as a simple polygon [13], a polygon with holes [15], a set of points [4], a set of line segments [6], along with different visibility models [8, 10].

Visibility graphs are used to describe real-world scenarios majority of which concern the mobile robots and path planning [2]. However, the physical limitations of the real world are usually overlooked or ignored while using visibility graphs. Since no camera, sensor, or guard (the objects represented by vertices of a visibility graph) has infinite range, two objects might not sense each other even though there are no obstacles in-between. Based on such a limitation, we assume that a pair of objects do not see each other if they are too far from each other. To model this notion, we adapt unit disk graphs.

G is called a *unit disk visibility graph* of $\mathcal{P}$ if the existence of an edge $uv \in E$ means that the straight line connecting p_u and p_v does not intersect any obstacles (e.g., some $p_w \notin \{p_u, p_v\}$), and the Euclidean distance between them is at most 1 unit. Unit disk point visibility graphs are well-defined by this definition whereas for unit disk segment graphs and polygon visibility graphs, the additional constraints are the following: *i*) the edges of a unit disk segment visibility graph cannot intersect any segment, *ii*) the edges of a unit disk polygon visibility graph must lie entirely inside the polygon, and *iii*) the segment lengths and the length of boundary edges of a polygon are at most 1 unit. We adapt vertex-visibility meaning that the visibility relations are between points and endpoints of the given set of geometric entities. Figure 1 shows examples of mentioned types of unit disk visibility graphs.

The *3-coloring* problem [9] is a famous NP-complete problem which asks if a graph has a (proper) 3-coloring, i.e. all vertices receive one of the three pre-given colors so that no two adjacent vertices receive the same color. In this paper, we show that this problem is NP-complete on unit disk visibility graphs of a set of line segments (even when every segment is axis-parallel, or when every segment has unit length), and a polygon with holes.

[1]Email: agaoglu@mail.muni.cz. This author has been supported by the Czech Science Foundation, project no. 20-04567S.
[2]Email: cagirici@ryerson.ca

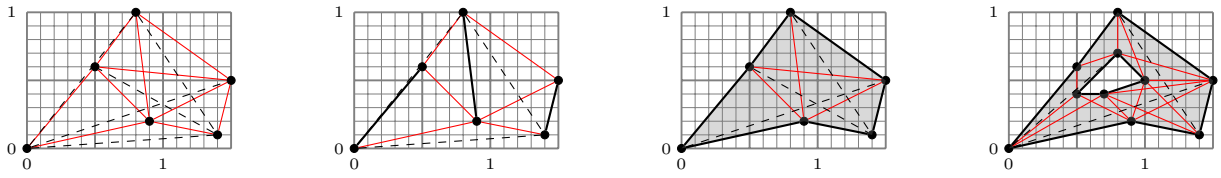


Figure 1: Unit disk visibility relations of (a) a set of points, (b) a set of line segments, (c) a simple polygon, and (d) a polygon with a hole.

2 Preliminary results

In this section, we prove that unit disk visibility graphs are not included in the (hierarchic) intersection of unit disk graphs and visibility graphs. We first show that visibility graphs are a proper subclass of unit disk visibility graphs. We assume that the given geometric set is a set of points, since every visibility graph considered in this paper can be embedded in the Euclidean plane; points, endpoints of a set of line segments, and the vertices of a polygon.

Lemma 1. *Consider a set $P = \{p_1, \ldots, p_n\}$ of points, and the visibility graph $G(P)$ of P. There exists an embedding of P, such that the Euclidean distance between every pair $p, q \in P$ is at most one unit, preserving the visibility relations.*

By Lemma 1, a given set P of points can be scaled down to obtain P' so that every point in P' is inside a unit circle, and the visibility graph $G(P)$ of P is exactly the same as the unit disk visibility graph of P'. We get the following.

Lemma 2. *If a problem $\mathfrak{Q}$ is NP-hard for point visibility graphs, then $\mathfrak{Q}$ is also NP-hard for unit disk point visibility graphs.*

We now obtain the following classification.

Lemma 3. *Unit disk visibility graphs are not a subclass of unit disk graphs.*

The idea used to prove Lemma 3 is that unit disk point, segment and polygon visibility graphs can contain an induced $K_{1,6}$ which is a forbidden induced subgraph for unit disk graphs [12].

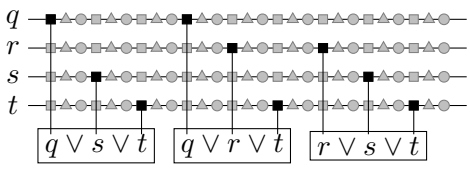
Lemma 4. *Unit disk graphs are a proper subclass of unit disk point visibility graphs, and* not *a subclass of unit disk segment and polygon visibility graphs.*

Proof sketch. Given a representation of a unit disk graph, we can simply perturb the disk centers slightly to obtain a set of points in general position in which no three points are collinear [7], which together with Lemma 3 shows that unit disk graphs are a proper subclass of unit disk point visibility graphs. However, unit disk segment visibility graphs require an even number of vertices since each segment has two endpoints, unit disk visibility graphs for simple polygons require Hamiltonian cycles forming the boundary of the simple polygon, and unit disk visibility graphs for polygons with holes require induced chordless cycles since visibility edges must lie within the polygon. Since these structures need not appear in unit disk graphs, unit disk segment and polygon visibility graphs are not a subclass of unit disk visibility graphs. □

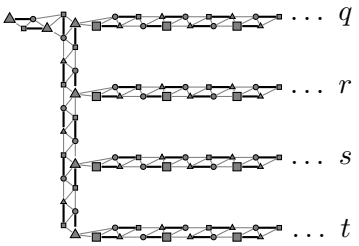
3 Main results

In this section, we mention our NP-hardness reductions. A polynomial-time NP-hardness reduction from an NP-complete problem $\mathfrak{Q}$ to another problem $\mathfrak{P}$ is to map any instance Φ of $\mathfrak{Q}$ to some instance

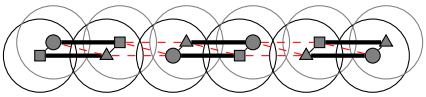
Ψ of $\mathfrak{P}$ such that Φ is a YES-instance of $\mathfrak{Q}$ if and only if Ψ is a YES-instance of $\mathfrak{P}$, in polynomial time and polynomial space. We first show that the 3-coloring problem for unit disk segment visibility graphs is NP-hard, using a reduction from the *Monotone NAE3SAT* problem which is a 3SAT variant [14] with no negated variables, and to satisfy the circuit, at least one true variable and one false variable must appear in each clause.



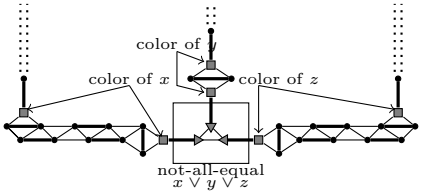
(a) An NAE3SAT formula with variables q, r, s, t, and the given clauses.



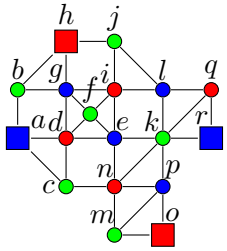
(b) The wires transferring one of two colors.



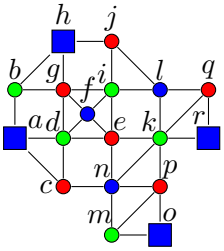
(c) A long edge gadget constructed by six line segments having a unique 3-coloring.



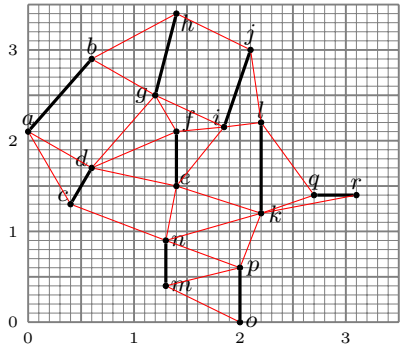
(d) The clause gadget for $x \vee y \vee z$.



(e) The edge crossing gadget transferring the color from a to r and h to o which has exactly two distinct proper 3-colorings.



(f) The same gadget in (e) and its only proper 3-coloring distinct from the 3-coloring given in (e) up to a permutation of colors.



(g) The embedding of (e).

Figure 2: The gadgets used in the proof of Theorem 5.

Theorem 5. *There is a polynomial-time reduction from the Monotone NAE3SAT problem to the 3-coloring problem for unit disk segment visibility graphs.*

Proof sketch. Given an NAE3SAT formula as in Figure 2a, three main components of our reduction are as follows.

(1) *A long edge* shown in Figure 2c is used to transfer a color from one end to the other (similar to transferring the truth assignment of a variable). This configuration, no matter how long, has a unique 3-coloring (up to permutation).

(2) *A clause gadget* shown in Figure 2d is modelled by three line segments xx', yy', and zz', not allowing three variables to have the same truth assignment. If all x, y, z have the same color, then this clause gadget cannot be 3-colored.

(3) *An edge crossing gadget* shown in Figure 2e describes a certificate for an edge crossing in the circuit so that it can be realized as a set of non-intersecting line segments with its embedding given in

Figure 2g. It has exactly two distinct 3-colorings (up to permutation) shown in Figure 2e and Figure 2f.

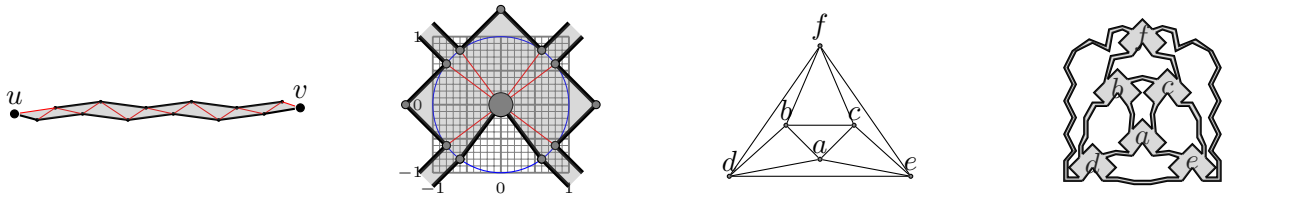
Given a Monotone NAE3SAT formula with m clauses $C_1, \ldots, C_m$ and n variables $q_1, \ldots, q_n$, we construct the corresponding unit disk segment visibility graph G as follows:

- For each variable q_i, add a vertex v_i to G and a long horizontal edge H_i transferring its color.
- For each clause C_i and each variable q_j in C_i, add a triangle T_i to G together with a long vertical edge V_j transferring the color of v_j.
- For each V_j crossing an H_i, add an edge crossing gadget to G replacing the intersection $V_j \cup H_i$.

Since this polynomial-time reduction works correctly, and the Monotone NAE3SAT problem is NP-complete [14], the 3-coloring problem for unit disk segment visibility graphs is also NP-complete. □

Remark 6. *The 3-coloring problem for unit disk graphs is NP-complete [11], and by Lemma 4, it is NP-complete for unit disk point visibility graphs. For an alternative reduction to [11], our gadgets can be utilized with small modifications.*

We now show that the 3-coloring problem for unit disk visibility graphs of polygons with holes is NP-hard by giving a reduction from the 3-coloring problem for 4-regular planar graphs [5].



(a) A corridor modelling the edges in a planar graph.

(b) The chamber that replaces the vertices.

(c) An example 4-regular planar graph.

(d) The polygon that corresponds to (c).

Figure 3: The gadgets used in the proof of Theorem 7.

Theorem 7. *There is a polynomial-time reduction from the 3-coloring problem for 4-regular planar graphs to the 3-coloring problem for unit disk visibility graphs of polygons with holes.*

Proof sketch. Given a 4-regular planar graph as in Figure 3c, two main components of our reduction are as follows.

(1) *A corridor* shown in Figure 3a replaces the edges. This gadget makes sure that the two ends of an edge receive different colors.

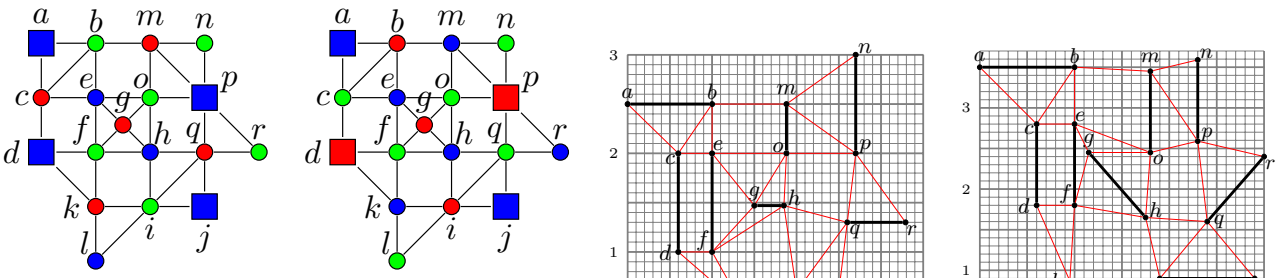
(2) *A chamber* shown in Figure 3b replaces the vertices. It is an induced subgraph with 12 vertices. One of them acts as *the central vertex*, and the boundary vertices act as *the openings* to the corridors which connect it to other chambers.

Given a 4-regular planar graph H on n vertices $v_1, \ldots, v_n$, we construct the corresponding polygon P with holes as follows:

- For each vertex v_i, add a chamber to P whose central vertex is vertex u_i.
- For each pair of adjacent vertices (v_i, v_j), add a corridor to P between the chambers with central vertices u_i and u_j.

Since this polynomial-time reduction works correctly, and the 3-coloring problem for 4-regular planar graphs is NP-complete [5], this problem is also NP-complete for unit disk visibility graphs of polygons with holes. □

We now consider two problems which were left open in [1], and show that the 3-coloring problem for unit disk segment visibility graphs is NP-hard even when we restrict every segment to be axis-parallel, and when we restrict every segment to have unit length, solving both open problems 1 and 2.



(a) The edge crossing gadget transferring the color from a to j and d to p which has exactly two distinct proper 3-colorings.

(b) The same gadget in (a) and its only proper 3-coloring distinct from the 3-coloring given in (a) up to a permutation of colors.

(c) The embedding of (a) where every segment is axis-parallel.

(d) The embedding of (a) where every segment has unit length.

Figure 4: The gadgets used in the proof of Theorem 8.

Theorem 8. *The 3-coloring problem for unit disk visibility graphs of line segments is NP-hard i) when every segment is axis-parallel, and ii) when every segment have unit length.*

Proof sketch. Firstly, we note here that both the long edge gadget shown in Figure 2c and the clause gadget shown in Figure 2d can be embedded in such a way that all their segments are both 1 unit long and parallel to either x-axis or y-axis. We replace the edge crossing gadget in Figure 2e used to prove the NP-completeness of 3-coloring problem for unit disk segment visibility graphs with the gadget shown in Figure 4a. By doing so, we describe a similar certificate for the circuit which has edge crossings (wires that overlap). Similar to the previous one, the new gadget also has exactly two distinct 3-colorings (up to permutation) shown in Figure 4a and Figure 4b. This gadget can be realized as a set of non-intersecting line segments where i) every segment is axis-parallel with its embedding given in Figure 4c, and ii) every segment has unit length with its embedding given in Figure 4d. □

Endpoints in Figure 2g	(x,y)
a	(0.00,2.10)
b	(0.60,2.90)
c	(0.40,1.30)
d	(0.60,1.70)
e	(1.40,1.50)
f	(1.40,2.10)
g	(1.20,2.50)
h	(1.40,3.40)
i	(1.85,2.15)
j	(2.10,3.00)
k	(2.20,1.20)
l	(2.20,2.20)
m	(1.30,0.40)
n	(1.30,0.90)
o	(2.00,0.00)
p	(2.00,0.60)
q	(2.70,1.40)
r	(3.10,1.40)

Endpoints in Figure 4c	(x,y)
a	(0.00, 2.50)
b	(1.00, 2.50)
c	(0.60, 2.00)
d	(0.60, 1.00)
e	(1.00, 2.00)
f	(1.00, 1.00)
g	(1.50, 1.47)
h	(1.85, 1.47)
i	(2.00, 0.60)
j	(2.50, 0.60)
k	(1.38, 0.42)
l	(1.38, 0.05)
m	(1.88, 2.50)
n	(2.70, 3.00)
o	(1.88, 2.00)
p	(2.70, 2.00)
q	(2.60, 1.30)
r	(3.29, 1.30)

Endpoints in Figure 4d	(x,y)
a	(0.00, 3.50)
b	(1.00, 3.50)
c	(0.60, 2.80)
d	(0.60, 1.80)
e	(1.00, 2.80)
f	(1.00, 1.80)
g	(1.15, 2.45)
h	(1.75, 1.65)
i	(1.90, 0.90)
j	(2.90, 0.90)
k	(0.95, 0.87)
l	(1.55, 0.07)
m	(1.80, 3.45)
n	(2.30, 3.59)
o	(1.80, 2.45)
p	(2.30, 2.59)
q	(2.40, 1.60)
r	(3.00, 2.40)

Table 1: The tables containing (x, y)-coordinates of the endpoints in Figure 2g, Figure 4c and Figure 4d.

4 Conclusions

We have introduced the unit disk visibility graphs, and proved the following:

- Visibility graphs are a proper subclass of unit disk visibility graphs.
- Unit disk graphs are a proper subclass of unit disk point visibility graphs while they are neither a subclass nor a superclass of unit disk visibility graphs of a set of line segments, simple polygons or polygons with holes.
- The 3-coloring problem for unit disk segment visibility graphs is NP-complete even when i) when every segment is axis-parallel, and when ii) when every segment has unit length.
- The 3-coloring problem for unit disk visibility graphs of polygons with holes is NP-complete.

We leave the 3-coloring problem for unit disk segment visibility graphs open when every segment is axis-parallel and also has unit length. In [3], it was proven that for visibility graphs of simple polygons, the 4-coloring problem can be solved in polynomial time, and the 5-coloring problem is NP-complete. We would like to investigate the complexity of the 3-coloring and 4-coloring problems for unit disk visibility graphs as future work.

References

[1] D. Ağaoğlu and O. Çağırıcı. "Unit Disk Visibility Graphs". In: *Extended Abstracts EuroComb 2021*. Ed. by J. Nešetřil, G. Perarnau, J. Rué, and O. Serra. Cham: Springer International Publishing, 2021, pp. 390–396.

[2] M. de Berg, O. Cheong, M. van Kreveld, and M. Overmars. *Computational geometry*. Third. Algorithms and applications. Springer-Verlag, Berlin, 2008, pp. xii+386. DOI: 10.1007/978-3-540-77974-2.

[3] O. Çağırıcı, P. Hliněný, and B. Roy. "On colourability of polygon visibility graphs". In: *37th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science*. Vol. 93. LIPIcs. Leibniz Int. Proc. Inform. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018, Art. No. 21, 14.

[4] J. Cardinal and U. Hoffmann. "Recognition and complexity of point visibility graphs". In: *31st International Symposium on Computational Geometry*. Vol. 34. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2015, pp. 171–185.

[5] D. P. Dailey. "Uniqueness of colorability and colorability of planar 4-regular graphs are NP-complete". In: *Discrete Math.* 30.3 (1980), pp. 289–293. DOI: 10.1016/0012-365X(80)90236-8.

[6] H. Everett, C. T. Hoàng, K. Kilakos, and M. Noy. "Planar segment visibility graphs". In: *Comput. Geom.* 16.4 (2000), pp. 235–243. DOI: 10.1016/S0925-7721(00)00009-2.

[7] G. D. da Fonseca, V. G. Pereira de Sá, R. C. S. Machado, and C. M. H. de Figueiredo. "On the recognition of unit disk graphs and the distance geometry problem with ranges". In: *Discrete Appl. Math.* 197 (2015), pp. 3–19. DOI: 10.1016/j.dam.2014.08.014.

[8] M. Garey, D. Johnson, and H. So. "An application of graph coloring to printed circuit testing". In: *16th Annual Symposium on Foundations of Computer Science*. Held in Berkeley, Calif., October 13–15, 1975. IEEE Computer Society, Long Beach, Calif., 1975, pp. 178–183.

[9] M. R. Garey and D. S. Johnson. *Computers and intractability*. A Series of Books in the Mathematical Sciences. A guide to the theory of NP-completeness. W. H. Freeman and Co., San Francisco, Calif., 1979, pp. x+338.

[10] M. Ghodsi, A. Maheshwari, M. Nouri-Baygi, J.-R. Sack, and H. Zarrabi-Zadeh. "α-visibility". In: *Comput. Geom.* 47.3, part A (2014), pp. 435–446. DOI: 10.1016/j.comgeo.2013.10.004.

[11] A. Gräf, M. Stumpf, and G. Weißenfels. "On coloring unit disk graphs". In: *Algorithmica* 20.3 (1998), pp. 277–293. DOI: 10.1007/PL00009196.

[12] M. V. Marathe, H. Breu, H. B. Hunt III, S. S. Ravi, and D. J. Rosenkrantz. "Simple heuristics for unit disk graphs". In: *Networks* 25.2 (1995), pp. 59–68. DOI: 10.1002/net.3230250205.

[13] J. O'Rourke. *Art gallery theorems and algorithms*. The Clarendon Press, Oxford University Press, New York, 1987, pp. xvi+282.

[14] T. J. Schaefer. "The complexity of satisfiability problems". In: *Conference Record of the Tenth Annual ACM Symposium on Theory of Computing (San Diego, Calif., 1978)*. ACM, New York, 1978, pp. 216–226.

[15] R. Wein, J. P. van den Berg, and D. Halperin. "The visibility-Voronoi complex and its applications". In: *Comput. Geom.* 36.1 (2007), pp. 66–87. DOI: 10.1016/j.comgeo.2005.11.007.

Polynomials for marked graphs and the chromatic symmetric function

José Aliste-Prieto[1*], Anna de Mier[2†], Rosa Orellana[3‡], José Zamora[4*]

[*]Departamento de Matemáticas, Facultad de Ciencias Exactas, Universidad Andres Bello, Chile
[†]Departament de Matemàtiques, Universitat Politècnica de Catalunya, Barcelona 08034, Spain
[‡]Mathematics Department, Dartmouth College, Hanover, NH 03755, U.S.A.

The full version of this work can be found in [1] and will be published elsewhere.

1 Introduction

In the mid 1990s Stanley [10] introduced a symmetric function generalization of the chromatic polynomial. For any graph $G = (V, E)$, if $x_1, x_2, \ldots$ are commuting variables then

$$X_G = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)} \ ,$$

where $v_1, \ldots, v_n$ are the vertices of G and the sum runs over all proper colorings $\kappa : V \to \mathbb{P}$ of G with positive integers. The function X_G is known as the *chromatic symmetric function* (CSF) of G. If we set $x_1 = x_2 = \ldots = x_k = 1$ and all other variables to zero, X_G specializes to the evaluation $\chi_G(k)$ of the ordinary chromatic polynomial. In this paper we allow graphs to have loops and multiple edges; as it is the case for the chromatic polynomial, the chromatic symmetric function of a graph with loops is zero, and removing duplicated edges from G does not change its chromatic symmetric function.

In his seminal paper Stanley expressed X_G using the classical bases of symmetric functions, proved many results and made several conjectures related to X_G. One open question is *Stanley's tree isomorphism problem* which asks whether the chromatic symmetric function distinguishes non-isomorphic trees (in [10] there is an example showing that it does not distinguish graphs with cycles). The answer is know to be positive for trees with fewer than 30 vertices, for caterpillars and for some smaller subclasses of trees (see for instance [2, 6, 7, 8]). In this paper we contribute to this problem by first finding a new expression for the CSF.

One of the results in Stanley's paper is the expansion of the CSF in the *power sum basis*: for any positive integer k let $p_k = \sum_{i \geq 1} x_i^k$ and for a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ of n let $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l}$. The family $\{p_\lambda \mid \lambda \vdash n\}$, that is, the collection of those p_λ where λ is a partition of n, is a basis for the homogeneous symmetric functions of degree n. Stanley [10, Theorem 2.5] proved that

$$X_G = \sum_{A \subseteq E} (-1)^{|A|} p_{\lambda(A)},$$

where $\lambda(A)$ is the partition of $|V|$ whose parts are the number of vertices in the connected components of the spanning subgraph $G|_A = (V, A)$.

Here we study the expansion of the CSF in another basis for symmetric functions that we call the star-basis. For this, we first need a result of Cho and van Willigenburg [3]. Let $\{G_k\}_{k \in \mathbb{N}}$ be a family of connected graphs with G_k having k vertices; given $\lambda \vdash n$ of length l, define $G_\lambda = G_{\lambda_1} \sqcup G_{\lambda_2} \sqcup \cdots \sqcup G_{\lambda_l}$.

[1]Email: jose.aliste@unab.cl. Research of JAP was partially supported by FONDECYT REGULAR 1160975 CONICYT Chile.

[2]Email: anna.de.mier@upc.edu. Research of AdM was partially supported by PID2020-113082GB-I00 funded by MCIN/AEI.

[3]Email: Rosa.C.Orellana@dartmouth.edu. Research of RO was partially supported by NSF grant DMS–300512.

[4]Email: josezamora@unab.cl. Research of JZ was partially supported by FONDECYT REGULAR 1180994 CONICYT Chile.

Then the set $\{X_{G_\lambda} \mid \lambda \vdash n\}$ is a basis for the homogeneous symmetric functions of degree n [3, Theorem 5]. The *star-basis* is the basis obtained by this construction when G_k is the star-graph $K_{1,k-1}$. We write $\mathfrak{st}_k = X_{K_{1,k-1}}$.

To develop our results we need to work with vertex-weighted graphs, and more generally with marked graphs, that are fit into the notion of graphs weighted by elements of a semigroup; we recall this concept in Section 2. There we also review the weighted chromatic symmetric function of Crew and Spirkl ([4]) and prove a relation, that we call deletion-near-contraction, that holds both for the weighted and ordinary CSF (as mentioned in Section 2, the ordinary CSF does not satisfy the classical deletion-contraction recurrence satisfied by the chromatic polynomial). This deletion-near-contraction relation can be used to recursively compute the expansion of the CSF in the star-basis. In Section 3 we introduce marked graphs and the marked graph polynomial, the M-polynomial, which is defined by deletion-contraction. We also consider a particular specialization of the M-polynomial that we call the D-polynomial. They both satisfy a deletion-near-contraction relation and the D-polynomial gives the expasion of the CSF in the star-basis upon a variable substitution. We also study when cancellations arise when doing these specialization and substitution. In the final section, we discuss the relation with the tree isomorphism problem and show that it has a positive answer for the class of trees of diameter at most five such that every vertex is either a leaf or adjacent to a leaf.

2 Semigroup-weighted graphs and the deletion-near-contraction formula

We start by reviewing the definition and operations of graphs weighted by a commutative semigroup $(S, \star)$, as in [5]. An S-weighted graph is a pair (G, s) where G is a graph and $\mathsf{s} : V \to S$ is a function. For a set of vertices $U = \{u_1, \ldots, u_k\}$ of G, its *total weight* is $\mathsf{s}(u_1) \star \cdots \star \mathsf{s}(u_k)$. The results in this section are for *weighted* graphs, that is, when $S = \mathbb{P}$ (the positive integers) and $\star$ is the usual integer addition; in this case, we use ω to denote the map s. However, we introduce some further definitions here for an arbitrary semigroup S since they are needed later in the paper.

Given s in S, we say that a multiset $\{\mathsf{s}_1, \mathsf{s}_2, \ldots, \mathsf{s}_l\}$ is a *partition* of s if $\mathsf{s}_1 \star \mathsf{s}_2 \star \cdots \star \mathsf{s}_l = \mathsf{s}$. An *$S$-partition* is a partition of s for some s in S. If $\lambda = \{\mathsf{s}_1, \mathsf{s}_2, \ldots, \mathsf{s}_l\}$ is an S-partition, then $\mathbf{z}_\lambda := z_{\mathsf{s}_1} z_{\mathsf{s}_2} \cdots z_{\mathsf{s}_l}$. The *$S$-partition* induced by A in (G, s), denoted by $\lambda(G, \mathsf{s}, A)$, is the S-partition determined by the total weights of the connected components of the spanning subgraph $(G|_A, \mathsf{s})$.

Given an edge $e = uv$ of G, the deletion and contraction operations extend naturally to (G, s). Note that the contraction operation is not defined for loops.

The *deletion of edge e* is the S-weighted graph $(G \setminus e, \mathsf{s})$ obtained by deleting the edge e from G and leaving s unchanged. The *contraction of the non-loop edge e* loop is the S-weighted graph $(G/e, \mathsf{s}/e)$, where G/e is the contraction of G by e into the *contracted vertex* v_e, and s/e is defined as

$$\mathsf{s}/e(u') = \begin{cases} \mathsf{s}(u) \star \mathsf{s}(v), & \text{if } u' = v_e, \\ \mathsf{s}(u'), & \text{otherwise.} \end{cases}$$

In [4], Crew and Spirkl extended the definition of the chromatic symmetric function to weighted graphs. The *weighted chromatic symmetric function* of a weighted graph (G, ω) is defined as

$$X_{(G,\omega)} = \sum_\kappa x_{\kappa(v_1)}^{\omega(v_1)} x_{\kappa(v_2)}^{\omega(v_2)} \cdots x_{\kappa(v_l)}^{\omega(v_l)},$$

where the sum ranges over all proper colorings κ of the vertices of G with positive integers. If (G, ω) is *unweighted*, that is, if $\omega(v) = 1$ for every vertex v, then we have that $X_G = X_{(G,\omega)}$. If e is an edge of G, then the weighted chromatic symmetric function satisfies the deletion-contraction formula:

$$X_{(G,\omega)} = X_{(G\setminus e,\omega)} - X_{(G/e,\omega/e)}. \tag{1}$$

If (G,ω) is simple, then $X_{(G,\omega)} = X_{G\setminus e,\omega} - X_{((G/e)^s,\omega/e)}$, where the superscript ^{s} denotes the simplification of the corresponding graph.

The main reason that the chromatic symmetric function does not satisfy the classical deletion-contraction relation is that when we contract an edge the resulting graph has one fewer vertex; therefore the resulting difference of the chromatic symmetric functions is no longer homogeneous. In contrast, when contracting an edge in a weighted graph the total weight is preserved (since the weights of the endpoints of the contracted edge are added), and thus the weighted chromatic symmetric functions of the original and the contracted graph are homogeneous of the same degree.

We now introduce a new relation that also preserves the number of vertices for unweighted graphs, and we will show that the chromatic symmetric function satisfies this relation. We will first define the *near-contraction operation* on weighted graphs: Given a weighted graph (G,ω) and a non-loop edge e in G, we denote by v' the contracted vertex in $(G/e,\omega/e)$ and define $(G,\omega)\odot e$ to be the weighted graph $(G/e\cup\{v'v''\},\omega')$ where v'' is a new vertex, and ω' is given by

$$\omega'(u) = \begin{cases} \omega/e(v') - 1, & \text{if } u = v', \\ 1, & \text{if } u = v'', \\ \omega/e(u), & \text{otherwise.} \end{cases}$$

In other words, $(G,\omega)\odot e$ is obtained by first contracting e into the contracted vertex v' and then attaching a leaf v'' of weight 1 to v' and decreasing the weight of v' by 1 to keep the total weight constant. The pendant edge $\ell_e = v'v''$ is called the *near-contracted edge* of $(G,\omega)\odot e$.

Proposition 1. *Let (G,ω) be a weighted graph and e be a non-loop edge of G. Then we have*

$$X_{(G/e,\omega/e)} = X_{((G,\omega)\odot e)\setminus\ell_e} - X_{(G,\omega)\odot e}. \tag{2}$$

Moreover, the weighted chromatic symmetric function satisfies the deletion near-contraction *formula:*

$$X_{(G,\omega)} = X_{(G\setminus e,\omega)} - X_{((G,\omega)\odot e)\setminus\ell_e} + X_{(G,\omega)\odot e}. \tag{3}$$

Moreover, if G is simple, then $X_{(G,\omega)} = X_{(G\setminus e,\omega)} - X_{((G,\omega)\odot e)^s\setminus\ell_e} + X_{((G,\omega)\odot e)^s}$.

Proof. It is easy to see that contracting the near-contracted edge ℓ_e in $(G,\omega)\odot e$ yields $(G/e,\omega/e)$ and thus the first assertion follows directly from the deletion-contraction formula (1) applied to $(G,\omega)\odot e$ and ℓ_e. Thus, (3) follows directly from substituting (2) into the deletion-contraction formula now applied to (G,ω) and e. Note that deleting or near-contracting an edge does not create loops, unless the near-contraction is applied to a parallel edge. Then, when G is simple all graphs that appear on the right-hand side of (3) are loopless, and the last statement follows since the weighted chromatic function of a loopless weighted graph equals that of its simplification. □

Remark 2. *Observe that if (G,ω) is unweighted, i.e., all the weights are equal to 1, then $(G\setminus e,\omega)$, $(G,\omega)\odot e$ and $((G,\omega)\odot e)\setminus\ell_e$ are also unweighted. Thus, Proposition 1 holds as well for the (unweighted) chromatic symmetric function.*

Note that if e is adjacent to a leaf of weight 1, then $(G,\omega)\odot e$ is weight-isomorphic to (G,ω) by an isomorphism sending e onto ℓ_e. In this case Equation (3) does not give any information. Thus, it only makes sense to apply Proposition 1 to edges that are either internal (i.e., not adjacent to leaves) or have an incident leaf of weight larger than one. Moreover, it is easy to see that deleting or near-contracting internal edges decreases the number of internal edges in a graph. Thus, after applying Proposition 1 recursively only to internal edges the procedure must always stop.

If G is a simple unweighted graph with no internal edges, then all the components of G are star graphs; we call such a graph a *star forest*. Thus, by recursively applying the deletion-near-contraction formula until no internal edges are left, we obtain the expansion of X_G in terms of the star-basis. In the next section we give another way of finding this expansion.

3 Marked graphs and marked-graph polynomials

Let $\mathbb{N}$ be the set of non-negative integers. A *mark* is a pair (w, d) where $w \in \mathbb{P}$, $d \in \mathbb{N}$ and $w \geq d+1$. We say that w is the *weight* of the mark and d is its *number of dots*. The set of all marks

$$\mathbb{M} = \{(w, d) \mid w \in \mathbb{P}, d \in \mathbb{N}, w \geq d+1\}$$

is a semigroup with the *dot-sum* operation, denoted $\dot{+}$, defined by

$$(w, d) \dot{+} (w', d') := (w + w', d + d' + 1).$$

Following the construction of Section 2, a *marked graph* is a graph with weights in the semigroup $(\mathbb{M}, \dot{+})$; we denote marked graphs by (G, m), where $\mathsf{m} : V \to \mathbb{M}$ is the map that assigns marks to vertices. If needed, we identify weighted graphs (that is, with weights in $\mathbb{P}$) with marked graphs with marks of the form $(w, 0)$, and we identify unweighted graphs with marked graphs with marks $(1, 0)$.

Definition 3. Let y and $\mathbf{z} = \{z_{w,d} \mid (w, d) \in \mathbb{M}\}$ be commuting indeterminates. The *marked graph polynomial* or *M-polynomial* of a marked graph (G, m) is defined by the following rules:

1. If G consists only of isolated vertices with corresponding marks (w_1, d_1),(w_2, d_2),...,(w_k, d_k), then
$$M_{(G,\mathsf{m})}(\mathbf{z}, y) = z_{w_1,d_1} z_{w_2,d_2} \cdots z_{w_k,d_k}.$$
2. If e is a loop of G, then $M_{(G,\mathsf{m})}(\mathbf{z}, y) = y M_{(G\setminus e,\mathsf{m})}(\mathbf{z}, y)$.
3. If e is a non-loop edge of G, then $M_{(G,\mathsf{m})}(\mathbf{z}, y) = M_{(G\setminus e,\mathsf{m})}(\mathbf{z}, y) + M_{(G/e,\mathsf{m}/e)}(\mathbf{z}, y)$.

The M-polynomial is a generalization of the W-polynomial of Noble and Welsh [9] to marked graphs. It is also a specialization of the **V**-polynomial of Ellis-Monaghand and Moffatt [5]. In particular, the following result, that gives the states-model expansion of the M-polynomial, follows from the corresponding formula for the **V**-polynomial [5, Theorem 3.5].

Proposition 4. *Let (G, m) be a marked graph with n vertices. Then, the M-polynomial is well-defined and it is a marked-isomorphism invariant. Moreover, it has the following* states model representation

$$M_{(G,\mathsf{m})}(\mathbf{z}, y) = \sum_{A \subseteq E(G)} \mathbf{z}_{\lambda(G,\mathsf{m},A)} (y-1)^{|A|-n+k_G(A)}, \tag{4}$$

where $k_G(A)$ is the number of connected components of $G|_A$ and $\lambda(G, \mathsf{m}, A)$ is the $\mathbb{M}$-partition induced by the total marks of the connected components of the spanning subgraph $G|_A$.

We next introduce a specialization of the M-polynomial that we call the D-polynomial. We will prove properties of this specialization that allow us to recover the star-expansion of the chromatic symmetric function. We use $\bullet_{w,d}$ to represent the one-vertex graph with mark (w, d).

Definition 5. The *D-polynomial* of a marked graph (G, m) is the polynomial in $\mathbb{Z}[\mathbf{z}, y]$ defined by

$$D_{(G,\mathsf{m})}(\mathbf{z}, y) = M_{(G,\mathsf{m})}\left(z_{w,d} = D_{\bullet_{w,d}}, y\right) \text{ with } D_{\bullet_{w,d}} = D_{\bullet_{w,d}}(\mathbf{z}, y) := \sum_{i=0}^{d} (-1)^i \binom{d}{i} z_{w-i,0} z_{1,0}^i.$$

Note that this definition is consistent since $M_{\bullet_{w,d}}(\mathbf{z}, y) = z_{w,d}$. In particular, note that for any weight w, $D_{\bullet_{w,0}} = z_{w,0}$.

Next, we extend the near-contraction operation to marked graphs. The graph $(G, \mathsf{m}) \odot e$ is obtained by first contracting $e = uv$ into the contracted vertex v_e, then attaching a leaf v' of weight $(1, 0)$, and then changing the mark of v_e to $(w_u + w_v - 1, d_u + d_v)$ where (w_u, d_u) and (w_v, d_v) are the marks of u and v respectively. With the same argument as in the proof of Proposition 1 one can show that the M-polynomial satisfies the deletion-near-contraction recurrence, and so also does the D-polynomial.

Proposition 6. *Let (G, m) be a marked graph and e a non-loop edge of G. Then the D-polynomial satisfies the following* deletion-contraction *and* deletion-near-contraction *relations:*

$$\begin{aligned} D_{(G,\mathsf{m})}(\mathbf{z}, y) &= D_{(G\setminus e,\mathsf{m})}(\mathbf{z}, y) + D_{(G/e,\mathsf{m}/e)}(\mathbf{z}, y), \\ D_{(G/e,\mathsf{m}/e)}(\mathbf{z}, y) &= D_{(G,\mathsf{m})\odot e}(\mathbf{z}, y) - D_{((G,\mathsf{m})\odot e)\setminus \ell_e}(\mathbf{z}, y), \\ D_{(G,\mathsf{m})}(\mathbf{z}, y) &= D_{(G\setminus e,\mathsf{m})}(\mathbf{z}, y) - D_{(G,\mathsf{m})\odot e\setminus \ell_e}(\mathbf{z}, y) + D_{(G,\mathsf{m})\odot e}(\mathbf{z}, y). \end{aligned}$$

Next we introduce another operation that deals with leaves of weight $(1, 0)$.

Definition 7. Let (G, m) be a marked graph. A vertex v is *absorbable* if v has degree one and mark $(1, 0)$. An edge e is *absorbable* if it is incident to an absorbable vertex v. If $e = uv$ is an absorbable edge where the vertex v is absorbable and $\mathsf{m}(u) = (w, d)$, the *absorption* of e is the marked graph $(G/e, \mathsf{m}')$, where the marks of all vertices of $V(G/e) \setminus \{u\}$ are the same as in G, and u has mark $\mathsf{m}'(u) = (w + 1, d)$. The *core* of (G, m) is the marked graph obtained after absorbing all the absorbable edges of (G, m).

The following result says that a marked graph and its core have the same D-polynomial. In particular, this reduces the computation of the D-polynomial of an unweighted tree to that of a weighted tree with fewer vertices. We omit the proof, which is by induction on the number of absorbable edges.

Theorem 8. *Let (G, m) be a marked graph and (K, m_K) be its core. Then*

$$D_{(G,\mathsf{m})}(\mathbf{z}, y) = D_{(K,\mathsf{m}_K)}(\mathbf{z}, y).$$

Theorem 8 implies that we will avoid many cancellations in the computation of the D-polynomial of a marked graph by first obtaining its core. However, some cancellations may still occur at some point if the core has some vertices with mark $(1, 0)$. By carefully analysing the sings of the terms in the D-polynomial we can identify when there are no cancellations.

Proposition 9. *Let (T, m) be a marked tree. The substitution $z_{w,d} = D_{\bullet_{w,d}}$ in the expression of the M-polynomial of (T, m) given by Proposition 4 yields a cancellation-free expression for the D-polynomial of (T, m) if and only all marks (w, d) of (T, m) satisfy $w > d + 1$.*

Finally, we show that the D-polynomial at $y = 0$ gives the expansion of X_G in the star basis. Note that if G is a tree, the D-polynomial does not have a term in y anyway. The proof is by induction by the number of internal edges.

Theorem 10. *Let G be a simple graph and (K, m_K) be its core. Then,*

$$X_G = D_{(K,\mathsf{m}_K)}(z_{w,0} = \mathfrak{st}_w, y = 0),$$

where $z_{w,0} = \mathfrak{st}_w$ means that we substitute each variable $z_{w,0}$ with $\mathfrak{st}_w$ for all $w \in \mathbb{P}$. In other words, the expansion of D into monomials encodes the chromatic star expansion of X_G.

4 The tree isomorphism question

As both M and D are polynomials for marked graphs, one may wonder whether there exist two marked trees, different up to marked-isomorphism, with the same M-polynomial. The answer is positive: let (P_1, m_1) be a 5-vertex path with marks, in order, $(2, 0), (1, 0), (2, 0), (3, 1), (1, 0)$, and let (P_2, m_2) be a 5-vertex path with marks, in order, $(1, 0), (2, 0), (1, 0), (3, 1), (2, 0)$. It follows by deletion-contraction applied to the second edge that both marked paths have the same M and D-polynomials.

By Theorem 8 the cores of (P_1, m_1) and (P_2, m_2) also have the same D-polynomial. However, the M-polynomials of the cores are different, since the states model representation implies that the respective highest-degree terms are $z_{4,1}z_{2,0}^2z_{1,0}$ and $z_{3,1}z_{3,0}z_{2,0}z_{1,0}$. This is an example of two different

M-polynomials leading to the same D-polynomial, the reason that the D-polynomials are equal is because of cancellations that occur when we apply the substitution $z_{w,d} = D_{\bullet_{w,d}}$.

We are interested in determining if the M and D polynomials distinguish unweighted trees, i.e., with all marks of the form $(1, 0)$. However, we work with weighted trees (i.e., with all marks of the form $(w, 0)$), because by Theorems 8 and 10 to compute the D-polynomial we use the core of the graph, which is a weighted graph, and from the D-polynomial we compute the chromatic symmetric function.

If there existed two non-isomorphic weighted trees (T_1, ω_1) and (T_2, ω_2) such that $D_{(T_1,\omega_1)} = D_{(T_2,\omega_2)}$, this would produce a pair of unweighted trees with the same chromatic symmetric function just by replacing each vertex of weight $(w, 0)$ with an unweighted vertex and $w - 1$ leaves adjacent to it. Hence, the problem of distinguishing unweighted trees with the chromatic symmetric function is equivalent to that of distinguishing weighted trees without absorbable leaves with the D-polynomial.

It is also unknown to the authors whether the M-polynomial distinguishes weighted trees. We remark that even if it was known that the M-polynomial distinguishes weighted trees up to weighted-isomorphism, this would not rule out the possibility of having a pair of non-isomorphic unweighted trees with the same chromatic symmetric function. In order to deduce this, one would need to show that the M-polynomial of a weighted tree can be recovered from its D-polynomial.

Our contribution to the tree isomorphism question is the following theorem. The proof is long and proceeds by carefully recovering some terms of the M-polynomial from the D-polynomial (which is equivalent to the CSF), the details can be found in [1]. The hypothesis that every vertex must be a leaf or adjacent to a leaf comes from Proposition 9.

Theorem 11. *Weighted trees of diameter at most 3 such that no vertex has weight* 1 *can be reconstructed from the D-polynomial.*

Thus, trees of diameter less or equal than 5 *all whose non-leaf vertices are adjacent to at least one leaf can be reconstructed from their chromatic symmetric functions.*

References

[1] J. Aliste-Prieto, A. de Mier, R. Orellana, and J. Zamora. *Marked graphs and the chromatic symmetric function*. Preprint. 2022. DOI: 10.48550/ARXIV.2202.11787.

[2] J. Aliste-Prieto and J. Zamora. "Proper caterpillars are distinguished by their chromatic symmetric function". In: *Discrete Math.* 315 (2014), pp. 158–164. DOI: 10.1016/j.disc.2013.10.016.

[3] S. Cho and S. van Willigenburg. "Chromatic bases for symmetric functions". In: *Electron. J. Combin.* 23.1 (2016), Paper 1.15, 7.

[4] L. Crew and S. Spirkl. "A deletion-contraction relation for the chromatic symmetric function". In: *European J. Combin.* 89 (2020), pp. 103143, 20. DOI: 10.1016/j.ejc.2020.103143.

[5] J. A. Ellis-Monaghan and I. Moffatt. "The Tutte-Potts connection in the presence of an external magnetic field". In: *Adv. in Appl. Math.* 47.4 (2011), pp. 772–782. DOI: 10.1016/j.aam.2011.02.004.

[6] S. Heil and C. Ji. "On an algorithm for comparing the chromatic symmetric functions of trees". In: *Australas. J. Combin.* 75 (2019), pp. 210–222.

[7] M. Loebl and J.-S. Sereni. "Isomorphism of weighted trees and Stanley's isomorphism conjecture for caterpillars". In: *Ann. Inst. Henri Poincaré D* 6.3 (2019), pp. 357–384. DOI: 10.4171/AIHPD/74.

[8] J. L. Martin, M. Morin, and J. D. Wagner. "On distinguishing trees by their chromatic symmetric functions". In: *J. Combin. Theory Ser. A* 115.2 (2008), pp. 237–253. DOI: 10.1016/j.jcta.2007.05.008.

[9] S. D. Noble and D. J. A. Welsh. "A weighted graph polynomial from chromatic invariants of knots". In: vol. 49. 3. Symposium à la Mémoire de François Jaeger (Grenoble, 1998). 1999, pp. 1057–1087.

[10] R. P. Stanley. “A symmetric function generalization of the chromatic polynomial of a graph”. In: *Adv. Math.* 111.1 (1995), pp. 166–194. DOI: 10.1006/aima.1995.1020.

An analogue of Chvátal's Hamiltonicity theorem for randomly perturbed graphs

Alexander Allin[1], Alberto Espuny Díaz[2]

Institut für Mathematik, Technische Universität Ilmenau, 98684 Ilmenau, Germany

The full version of this work will be published elsewhere. This research was supported by the Carl-Zeiss-Foundation and by DFG grant PE 2299/3-1.

Abstract

We consider Hamilton cycles in randomly perturbed graphs, that is, graphs obtained as the union of a deterministic graph H and a random graph $G(n,p)$. While most research into randomly perturbed graphs assumes a minimum degree condition on H, here we consider conditions on its degree sequence. Under the assumption of a degree sequence of H which is comparable with the classical condition of Chvátal (dependent on a parameter α analogous to the minimum degree condition in typical results in the area), we prove that there exists some constant $C = C(\alpha)$ such that taking $p = C/n$ suffices to a.a.s. obtain a Hamilton cycle in $H \cup G(n,p)$. Our result is best possible both in terms of the degree sequence condition and the asymptotic value of p, and extends the known results about Hamiltonicity in randomly perturbed graphs. We also provide results about pancyclicity under the same conditions.

1 Introduction

We call a graph Hamiltonian if it contains a spanning cycle, that is, a cycle containing all vertices of the graph. The study of Hamilton cycles has become a central area in graph theory. The question whether a graph is Hamiltonian was shown to be NP-complete by Karp [7], and so many results in the area provide sufficient conditions for the existence of such cycles. One such condition is given by a classical result of Dirac [5], which establishes that any graph on $n \geq 3$ vertices with minimum degree at least $n/2$ must be Hamiltonian. This result is optimal: the minimum degree condition cannot be improved. However, Dirac's theorem can be generalised when the degree sequence is taken into account. The degree sequence of a graph is the monotonic non-decreasing sequence of its vertex degrees. In this direction, Pósa [9] proved the following. Let H be an n-vertex graph with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ such that for every $i < n/2$ we have $d_i \geq i+1$. Then, H is Hamiltonian. This allows roughly half of the vertices to have degrees below what is required by Dirac's theorem. Pósa's result was further improved by Chvátal [3], who showed that, if for all $i < n/2$ the more general condition $d_i \geq i+1$ or $d_{n-i} \geq n-i$ holds, then H is Hamiltonian. This result is best possible in the following sense. Suppose a sequence $a_1 \leq \cdots \leq a_n$ violates the condition for any $i < n/2$. Then, there exists a non-Hamiltonian graph H with degree sequence $d_1 \leq \cdots \leq d_n$ such that $d_i \geq a_i$ for all i.

Hamiltonicity has also been extensively studied in the context of the binomial random graph. The binomial random graph $G(n,p)$ is an n-vertex random graph in which all of the $\binom{n}{2}$ possible edges are independently present with probability p. In this context, Pósa [10] proved that taking $p = \Omega(\log n/n)$ suffices to ensure that asymptotically almost surely (a.a.s.), i.e., with probability tending to 1 as n tends to infinity, $G(n,p)$ is Hamiltonian. This cannot be improved, as taking $p = o(\log n/n)$ ensures that a.a.s. $G(n,p)$ will have isolated vertices.

[1]Email: alexander.allin@tu-ilmenau.de.
[2]Email: alberto.espuny-diaz@tu-ilmenau.de.

In this paper we will focus on randomly perturbed graphs. This model, first introduced by Bohman, Frieze and Martin [1], considers the union of a deterministic graph H with a binomial random graph $G(n,p)$ on the same vertex set. Bohman, Frieze and Martin [1] proved that for any $\alpha > 0$ there exists a constant $C > 0$ such that, for every n-vertex graph H with minimum degree at least αn, a.a.s. the graph $H \cup G(n, C/n)$ is Hamiltonian. Observe how the necessary p is improved by a logarithmic factor with respect to the result of Pósa [10]. Since then, randomly perturbed graphs have attracted a considerable research interest. A wealth of references concerning Hamilton cycles and other spanning structures in randomly perturbed graphs can be found, e.g., in [6]. The original result of Bohman, Frieze and Martin was expanded upon by Krivelevich, Kwan and Sudakov [8], who reproved the result using different methods and generalised it to hypergraphs. Very recently, Hahn-Klimroth, Maesaka, Mogge, Mohr and Parczyk [6] optimised the dependency of C on α, which allowed them to extend the result to $\alpha = o(1)$. In all three cases, a condition on the minimum degree of the deterministic graph is needed. Therefore, these results can be regarded as analogues of Dirac's theorem in the setting of randomly perturbed graphs.

In our paper we shall study the Hamiltonicity of $H \cup G(n,p)$ in the case that the degree sequence of the deterministic graph H satisfies conditions similar to those in Chvátal's theorem. To be precise, we will consider the following condition.

Definition 1. Given integers $n, k > 0$, an (n,k)-*Chvátal graph* is an n-vertex graph with degree sequence $d_1 \leq \cdots \leq d_n$ such that, for every $i < n/2$,

$$d_i \geq \min\{i+1, k\} \qquad \text{or} \qquad d_{n-i} \geq n - i.$$

For graphs satisfying the condition in Definition 1, we obtain the following result. This can be regarded as an analogue of Chvátal's theorem for randomly perturbed graphs.

Theorem 2. *For any $\alpha > 0$ there exists a constant $C > 0$ such that, for any $(n, \alpha n)$-Chvátal graph H, a.a.s. the graph $H \cup G(n, C/n)$ is Hamiltonian.*

This result implies the result of Bohman, Frieze and Martin [1] and is best possible in two ways. First, the lower bound on p for the random graph is optimal up to the constant factor, as follows from the aforementioned result. Second, the degree condition given in Definition 1 cannot be improved. Indeed, let ℓ be a fixed index and consider the n-vertex graph H formed by the union of two complete graphs $K_{\ell+1}$ and $K_{n-\ell}$ intersecting in precisely one vertex. Let $A \subseteq V(H)$ be the set of all ℓ vertices of degree ℓ, B the set of all $n-\ell-1$ vertices of degree $n-\ell-1$, and v the only vertex of degree $n-1$. Writing $d_1 \leq \cdots \leq d_n$ for the degree sequence of H, we observe that, for every $i < n/2$ with $i \neq \ell$, the condition $d_i \geq i+1$ is satisfied. However, $d_\ell = \ell$ and $d_{n-\ell} = n-\ell-1$, so H does not satisfy the condition in Definition 1. Note, though, that the condition is almost satisfied. Now consider the perturbed graph $H \cup G(n,p)$. If we condition on the event $\mathcal{E}$ that $E(A,B) = \varnothing$ in $G(n,p)$, then $H \cup G(n,p)$ will not be 2-connected, as deleting v disconnects it. In particular, this means that $H \cup G(n,p)$ cannot be Hamiltonian. Thus, in order for $H \cup G(n,p)$ to be Hamiltonian a.a.s., we must have $\mathbb{P}[\mathcal{E}] = o(1)$. However, for any constant $C > 0$ and n sufficiently large, the probability that $\mathcal{E}$ holds is

$$\mathbb{P}[E(A,B) = \varnothing] = \left(1 - \frac{C}{n}\right)^{\ell(n-\ell-1)} \geq \mathrm{e}^{-2C\ell},$$

which does not depend on n. Hence, $H \cup G(n, C/n)$ will not a.a.s. be Hamiltonian.

A property generalising Hamiltonicity is pancyclicity. We call an n-vertex graph *pancyclic* if, for every $3 \leq \ell \leq n$, it contains a cycle of length ℓ. In 1971, Bondy [2] proved that any n-vertex graph with minimum degree strictly larger than $n/2$ must be pancyclic. Schmeichel and Hakimi [11] later showed that, if a graph satisfies Chvátal's degree sequence condition, then it is either pancyclic or bipartite. In the random setting, Cooper and Frieze [4] proved that the threshold for pancyclity in $G(n,p)$ is the same as that of Hamiltonicity. In the context of randomly perturbed graphs, Krivelevich, Kwan and

Sudakov [8] also considered pancyclicity and obtained a result analogous to the result of Bohman, Frieze and Martin [1] for Hamiltonicity. That is, given $\alpha > 0$, they proved that there exists some $C > 0$ such that, given any graph H with minimum degree at least αn, a.a.s. the graph $H \cup G(n, C/n)$ is pancyclic. This can be seen as an analogue of the result of Bondy for randomly perturbed graphs. We extend the result of [8] to give a corresponding analogue of Schmeichel and Hakimi's result.

Theorem 3. *For any $\alpha > 0$ there exists some $C > 0$ such that, for any $(n, \alpha n)$-Chvátal graph H, a.a.s. the graph $H \cup G(n, C/n)$ is pancyclic.*

Observe that Theorem 3 implies Theorem 2, and is therefore best possible in the same sense as the latter. We will obtain Theorem 3 directly from Theorem 2 and the following general result.

Theorem 4. *If an n-vertex graph H is Hamiltonian, then a.a.s. $H \cup G(n, 4 \log n/e(H))$ is pancyclic.*

Recall that Cooper and Frieze [4] showed that $p = \Omega(\log n/n)$ is needed to ensure that a.a.s. the random graph $G(n, p)$ is pancyclic. Thus, as soon as $e(H) = \omega(n)$, Theorem 4 improves upon the purely random result. Moreover, for our proof of Theorem 3, we are particularly interested in dense graphs H. In these cases, our result guarantees that adding the very sparse graph $G(n, \Theta(\log n/n^{-2}))$ to H is sufficient to a.a.s. ensure pancyclity.

2 Hamiltonicity

Theorem 2 will follow from the following deterministic result. This is similar to [8, Theorem 3] and extends this result to a degree-sequence setting.

Lemma 5. *Let H be an $(n, 2k)$-Chvátal graph and suppose that for any disjoint sets $A, B \subseteq V(H)$ of size $|A|, |B| \geq k$ there is an edge $ab \in E(H)$ with $a \in A$ and $b \in B$. Then, H is Hamiltonian.*

Proof. We proceed by contradiction. Assume that the statement does not hold and let H be a maximal counterexample, so that adding any new edge creates a Hamilton cycle. Select a pair of vertices $v, w \in V(H)$ with $vw \notin E(H)$ which maximises $d(v) + d(w)$. Note that, by the maximality of H, there must be a Hamiltonian path P having v and w as endpoints. Taking v as the starting point, the path P imposes an ordering on $V(H)$. Given $x, y \in V(H)$, let us write $x < y$ if x precedes y on P. Given a vertex $u \in V(H) \setminus \{v\}$, we shall write u^- for its immediate predecessor on P; similarly, given $u \in V(H) \setminus \{w\}$, we will write u^+ for its immediate successor. For a set $S \subseteq V(H) \setminus \{v\}$, we will write $S^- \coloneqq \{s^- : s \in S\}$, and for $S \subseteq V(H) \setminus \{w\}$ we define $S^+ \coloneqq \{s^+ : s \in S\}$. We will consider two cases.

Case 1. Assume that $d(v), d(w) \geq 2k$. Let V_1 be the set of the first k neighbours of v in the ordering from above and let V_2 be the set of the next k neighbours. Split the first $2k$ neighbours of w into sets W_1 and W_2 analogously.

First, let us suppose that for every vertex $v_1 \in V_1$ and every $w_2 \in W_2$ we have $v_1 \leq w_2$. The sets V_1^- and W_2^+ are then disjoint sets of size k. Thus, by assumption, there is at least one edge $e = v_1^- w_2^+ \in E(H)$ with $v_1 \in V_1$ and $w_2 \in W_2$. But now we can construct a Hamilton cycle by starting at v, following P until v_1^-, using e to w_2^+, following P until w, going to w_2, following P in the opposite direction until v_1 and, finally, returning to v.

Therefore, we may suppose that there are $v_1 \in V_1$ and $w_2 \in W_2$ such that $w_2 < v_1$. Note that, by the way we chose V_1, V_2 and W_1, W_2, we must therefore have $w_1 < v_2$ for every $v_2 \in V_2$ and every $w_1 \in W_1$. Now V_2^- and W_1^- are disjoint sets of size k, so there is an edge $e = v_2^- w_1^- \in E(H)$, with $v_2 \in V_2$ and $w_1 \in W_1$, between them. Then, we can obtain a Hamilton cycle as follows: start at v and follow P until w_1^-, use the edge e to v_2^-, follow P in the opposite direction up to w_1, go to w, and follow P to v_2, from where we can return to v.

Case 2. Assume now that $d(v) < 2k$ or $d(w) < 2k$. Without loss of generality we may suppose that $d(v) < 2k$. If $N(v)^-$ and $N(w)$ intersect, then there is a Hamilton cycle. Indeed, let $u \in N(v)^- \cap N(w)$.

To build a Hamilton cycle, start at v and follow P to u, go to w, and follow P in the opposite direction until u^+, from where we can return to v. So we may assume that $N(v)^-$ and $N(W)$ are disjoint. Observing that w cannot be in $N(v)^-$ nor $N(w)$, we must have $d(v) + d(w) = |N(v)^-| + |N(w)| < n$. Writing $h \coloneqq d(v)$, it follows that $d(w) < n - h$. Moreover, note that, by the maximality of $d(v) + d(w)$, every vertex in $N(v)^-$ must have degree at most h. This gives at least h vertices of degree at most h. Considering the degree sequence $d_1 \leq \cdots \leq d_n$ of H, we hence have $d_h \leq h$.

As $h < 2k$ and since H is $(n, 2k)$-Chvátal, we must have $d_{n-h} \geq n - h$. But then there are at least $h + 1$ vertices of degree at least $n - h$, of which at most h can be neighbours of v. So there must be some $u \in V(H)$ with $d(u) \geq n - h$ such that $uv \notin E(H)$. But then the pair v, u violates the maximality of $d(v) + d(w)$, as $d(v) + d(w) < n \leq d(v) + d(u)$. □

To complete the proof of Theorem 2, it only remains to show that a.a.s. the randomly perturbed graph $H \cup G(n, p)$ will satisfy the conditions of Lemma 5.

Proof of Theorem 2. We claim that, if $C \geq 8\alpha^{-2}$, then the random graph $G(n, C/n)$ a.a.s. satisfies the following: for every pair A, B of disjoint sets of vertices of size at least $\alpha n/2$, we have $E(A, B) \neq \varnothing$. Indeed, let us fix such a pair A, B and write $p \coloneqq C/n$. Then, we have

$$\mathbb{P}[E(A, B) = \varnothing] = (1 - p)^{|A||B|} \leq (1 - p)^{\alpha^2 n^2/4} \leq \mathrm{e}^{-p\alpha^2 n^2/4} = \mathrm{e}^{-C\alpha^2 n/4}.$$

Furthermore, there are at most 2^{2n} choices for A and B. Therefore, the probability that there is such a pair A and B with $E(A, B) = \varnothing$ is at most $2^{2n}\mathrm{e}^{-C\alpha^2 n/4} \leq (2/\mathrm{e})^{2n}$, which tends to 0.

For this C, we therefore know that a.a.s. $H \cup G(n, C/n)$ satisfies the conditions of Lemma 5 with $k = \alpha n/2$. Hence, a.a.s. $H \cup G(n, C/n)$ contains a Hamilton cycle. □

3 Pancyclicity

Let us now turn our attention to pancyclicity. We will first give a proof of Theorem 4 and use this to show Theorem 3. For the proof of Theorem 4 we will first need the following lemma.

Lemma 6. *Let H be an n-vertex Hamiltonian graph. Then, for each $\ell \in \{3, \dots, n-1\}$ there is a set S_ℓ of size $|S_\ell| \geq e(H)/2$ such that, for any $e \in S_\ell$, the graph $H + e$ contains a cycle of length ℓ.*

Proof. Let C be a Hamilton cycle in H and label the vertices of H as $v_1, v_2, \dots, v_n$ along C. In the following we will consider the indices modulo n, so $v_i = v_{i+n}$.

We will first consider cycles of length $\ell \leq (n + 3)/2$. Fix a vertex $v_i \in V(H)$ and note that $P_1 \coloneqq v_i v_{i+1} \dots v_{i+\ell-2}$ and $P_2 \coloneqq v_i v_{i-1} \dots v_{i-\ell+2}$ are paths of length $\ell - 2$ in H. As $\ell \leq (n + 3)/2$, the paths P_1 and P_2 are disjoint apart from v_i. Therefore, any neighbour w of v_i is contained in at most one of the paths. If it is not contained in P_1, then the edge $e = wv_{i+\ell-2}$ closes an ℓ-cycle; if it is not contained in P_2, then so does $e = wv_{i-\ell+2}$. Let $S_\ell^i \subseteq E(K_n)$ be the set of all such edges e that close an ℓ-cycle in this manner, over all neighbours w of v_i. By the observation above we must have $|S_\ell^i| \geq d(v_i)$.

Now suppose $v_j v_k \in S_\ell^i$ for some $i \in \{1, \dots, n\}$. Then, by definition, we must have $j = i + \ell - 2$, $j = i - \ell + 2$, $k = i + \ell - 2$ or $k = i - \ell + 2$. In particular, for any given edge $e \in E(K_n)$ there are at most four distinct $i \in \{1, \dots, n\}$ such that $e \in S_\ell^i$. Now let $S_\ell \coloneqq \bigcup_{i=1}^n S_\ell^i$. Since each edge appears in at most four of the S_ℓ^i, it follows that $|S_\ell| \geq \frac{1}{4}\sum_{i=1}^n d(v_i) = e(H)/2$, as needed.

Now we consider cycles of length $\ell > (n + 3)/2$. Fix a vertex v_i. We are interested in the paths $P_1 \coloneqq v_i v_{i+1} \dots v_{i+\ell-1}$ and $P_2 \coloneqq v_i v_{i-1} \dots v_{i-\ell+1}$, which have length $\ell - 1$. Note that every vertex is contained in at least one of those paths. Now let v_m be a neighbour of v_i and suppose that v_m lies on P_1. Then, $e = v_{m-1}v_{i+\ell-1}$ closes an ℓ-cycle as follows: start at v_i and follow P_1 until v_{m-1}, use e to arrive at $v_{i+\ell-1}$, continue on P in the opposite direction up to v_m, and return to v. Similarly, if v_m lies on P_2, then $e = v_{m+1}v_{i-\ell+1}$ closes an ℓ-cycle. As before, let S_ℓ^i be the set of all these edges e over all neighbours v_m of v_i, so $|S_\ell^i| \geq d(v_i)$. Again, if $v_j v_k \in S_\ell^i$, then we must have $j = i + \ell - 1$, $j = i - \ell + 1$,

$k = i + \ell - 1$ or $k = i - \ell + 1$. Thus, any edge of $E(K_n)$ is contained in at most four sets S_ℓ^i. Writing $S_\ell \coloneqq \bigcup_{i=1}^n S_\ell^i$ we conclude that $|S_\ell| \geq \frac{1}{4}\sum_{i=1}^n d(v_i) = e(H)/2$, by the same double counting argument as above. □

With this, we can now prove Theorem 4

Proof of Theorem 4. Suppose that H is a Hamiltonian graph. By Lemma 6, for each $\ell \in \{3, \dots, n-1\}$ there exists a set $S_\ell \subseteq E(K_n)$ such that $|S_\ell| \geq e(H)/2$ and, for every $e \in S_\ell$, the graph $H + e$ contains an ℓ-cycle. Let $p \geq 4\log n/e(H)$ and consider the random graph $G(n,p)$ on the same vertex set as H. The probability that no edge from S_ℓ appears in $G(n,p)$ is

$$\mathbb{P}\left[E\left(G\left(n,p\right)\right) \cap S_\ell = \varnothing\right] = (1-p)^{|S_\ell|} \leq (1-p)^{e(H)/2} \leq \mathrm{e}^{-pe(H)/2} \leq \mathrm{e}^{-2\log n}.$$

Taking the union bound over all S_ℓ, the probability that there is at least one $\ell \in \{3, \dots, n-1\}$ such that $G(n,p)$ contain no edge of S_ℓ is at most $n\mathrm{e}^{-2\log n} = n^{-1}$, which tends to 0. Therefore, a.a.s. $G(n,p)$ contains at least one edge out of every set S_ℓ. By our choice of the sets S_ℓ, a.a.s. $H \cup G(n, 4\log n/e(H))$ contains an ℓ-cycle for every $\ell \in \{3, \dots, n\}$, as required. □

Now Theorem 3 follows as a simple corollary of Theorem 2 and Theorem 4.

Proof of Theorem 3. By Theorem 2, we first find a C_1 such that a.a.s. $H' = H \cup G(n, C_1/n)$ is Hamiltonian. Let us condition on this event. Then, by Theorem 4, a.a.s. $H' \cup G(n,p)$ with $p = 4\log n/e(H')$ is pancyclic. Since H is $(n, \alpha n)$-Chvátal, so is H'. Therefore, at least half of the vertices of H' have degree at least αn and, thus, $e(H') \geq \alpha n^2/4$. Hence, for $C_2 \geq 16\alpha^{-1}$, a.a.s. $H \cup G(C_1/n) \cup G(n, C_2 \log n/n^2)$ is pancyclic. However note that in $G(C_1/n) \cup G(n, C_2\log n/n^2)$ every edge is present with probability at least $C_1/n + C_2 \log n/n^2$. Note that the latter is less than C/n for any $C = C_1 + \epsilon$ with $\epsilon > 0$ and sufficiently large n. Hence, we can conclude that for such a C, a.a.s. $H \cup G(n, C/n)$ will be pancyclic. □

References

[1] T. Bohman, A. Frieze, and R. Martin. "How many random edges make a dense graph Hamiltonian?" In: *Random Structures Algorithms* 22.1 (2003), pp. 33–42. DOI: 10.1002/rsa.10070.

[2] J. A. Bondy. "Pancyclic graphs. I". In: *J. Combinatorial Theory Ser. B* 11 (1971), pp. 80–84. DOI: 10.1016/0095-8956(71)90016-5.

[3] V. Chvátal. "On Hamilton's ideals". In: *J. Combinatorial Theory Ser. B* 12 (1972), pp. 163–168. DOI: 10.1016/0095-8956(72)90020-2.

[4] C. Cooper and A. M. Frieze. "Pancyclic random graphs". In: *Random graphs '87 (Poznań, 1987)*. Wiley, Chichester, 1990, pp. 29–39.

[5] G. A. Dirac. "Some theorems on abstract graphs". In: *Proc. London Math. Soc. (3)* 2 (1952), pp. 69–81. DOI: 10.1112/plms/s3-2.1.69.

[6] M. Hahn-Klimroth, G. S. Maesaka, Y. Mogge, S. Mohr, and O. Parczyk. "Random perturbation of sparse graphs". In: *Electron. J. Combin.* 28.2 (2021), Paper No. 2.26, 12. DOI: 10.37236/9510.

[7] R. M. Karp. "Reducibility among Combinatorial Problems". In: *Complexity of Computer Computations*. The IBM Research Symposia Series. Springer, Boston, MA, 1972, pp. 85–103. DOI: 10.1007/978-1-4684-2001-2_9.

[8] M. Krivelevich, M. Kwan, and B. Sudakov. "Cycles and matchings in randomly perturbed digraphs and hypergraphs". In: *Combin. Probab. Comput.* 25.6 (2016), pp. 909–927. DOI: 10.1017/S0963548316000079.

[9] L. Pósa. “A theorem concerning Hamilton lines”. In: *Magyar Tud. Akad. Mat. Kutató Int. Közl.* 7 (1962), pp. 225–226.

[10] L. Pósa. “Hamiltonian circuits in random graphs”. In: *Discrete Math.* 14.4 (1976), pp. 359–364. DOI: 10.1016/0012-365X(76)90068-6.

[11] E. F. Schmeichel and S. L. Hakimi. “Pancyclic graphs and a conjecture of Bondy and Chvátal”. In: *J. Combinatorial Theory Ser. B* 17 (1974), pp. 22–34. DOI: 10.1016/0095-8956(74)90043-4.

On sumsets of nonbases of maximum size

Béla Bajnok[1*], Péter Pál Pach[2†,‡]

[*]Department of Mathematics, Gettysburg College, Gettysburg, PA 17325, USA
[†]Department of Computer Science and Information Theory, Budapest University of Technology and Economics, Műegyetem rkp. 3., H-1111 Budapest, Hungary.
[‡]MTA-BME Lendület Arithmetic Combinatorics Research Group, ELKH, Műegyetem rkp. 3., H-1111 Budapest, Hungary.

The full version of this work will be published elsewhere.

Abstract

Let G be a finite abelian group. A nonempty subset A in G is called a basis of order h if $hA = G$; when $hA \neq G$, it is called a nonbasis of order h. Our interest is in all possible sizes of hA when A is a nonbasis of order h in G of maximum size; we provide the complete answer when $h = 2$ or $h = 3$.

1 Introduction

Let G be a finite abelian group of order $n \geq 2$, written in additive notation. For a positive integer h, the *Minkowski sum* of nonempty subsets $A_1, \ldots, A_h$ of G is defined as

$$A_1 + \cdots + A_h = \{a_1 + \cdots + a_h \ : \ a_1 \in A_1, \ldots, a_h \in A_h\}.$$

When $A_1 = \cdots = A_h = A$, we simply write hA, which then is the collection of sums of h not-necessarily-distinct elements of A.

We say that a nonempty subset A of G is *h-complete* (alternatively, a *basis of order h*) if $hA = G$; while, if hA is a proper subset of G, we say that A is *h-incomplete*. The *h-critical number* $\chi(G, h)$ of G is defined as the smallest positive integer m for which all m-subsets of G are h-complete; that is:

$$\chi(G, h) = \min\{m \ : \ A \subseteq G, |A| \geq m \Rightarrow hA = G\}.$$

It is easy to see that for all G and h we have $hG = G$, so $\chi(G, h)$ is well defined. The value of $\chi(G, h)$ is now known for every G and h—see [1, 2].

The following question then arises naturally: What can one say about the size of hA if A is an h-incomplete subset of maximum size in G? Namely, we aim to determine the set

$$S(G, h) = \{|hA| \ : \ A \subset G, \ |A| = \chi(G, h) - 1, \ hA \neq G\}.$$

In this paper we attain the complete answer to this question for $h = 2$ and $h = 3$. For $h = 2$, we find that the situation is greatly different for groups of even and odd order.

Theorem 1. *Let G be an abelian group of order n.*

[1]Email: bbajnok@gettysburg.edu.
[2]Email: ppp@cs.bme.hu. Research of P. P. P. is supported by the Lendület program of the Hungarian Academy of Sciences (MTA) and by the National Research, Development and Innovation Office NKFIH (Grant Nr. K124171 and K129335).

1. *When n is even, the maximum size of a 2-incomplete subset of G is $n/2$, and the elements of $S(G,2)$ are of the form $n - n/d$ where d is some even divisor of n; in fact all such integers are possible, with the exception that $3n/4$ arises only when the exponent of G is divisible by 4.*

2. *When n is odd, the maximum size of 2-incomplete subsets of G is $(n-1)/2$; furthermore, when G is of order 3, 5, or is noncyclic and of order 9, then $S(G,2) = \{n-2\}$, and for all other groups of odd order we have $S(G,2) = \{n-2, n-1\}$.*

For $h = 3$ we separate three cases.

Theorem 2. *Let G be an abelian group of order n.*

1. *When n has prime divisors congruent to 2 mod 3, and p is the smallest such prime, the maximum size of a 3-incomplete subset is $(p+1)n/(3p)$, and we have $S(G,3) = \{n - n/p\}$.*

2. *When n is divisible by 3 but has no divisors congruent to 2 mod 3, then the maximum size of a 3-incomplete subset is $n/3$, and the elements of $S(G,3)$ are of the form $n - n/d$ or $n - 2n/d$ where d is some divisor of n that is divisible by 3; furthermore, all such integers are possible, with the exceptions of $2n/3$ and $n - 2n/d$ when the highest power of 3 that divides d is more than the highest power of 3 that divides the exponent of G.*

3. *In the case when all divisors of n are congruent to 1 mod 3, then the maximum size of a 3-incomplete subset is $(n-1)/3$, and $S(G,3) = \{n-3, n-1\}$, unless G is an elementary abelian 7-group, in which case $S(G,3) = \{n-3\}$.*

We should note that the three cases addressed in Theorem 2 are the same as those used while studying sumfree sets—see [3] and [4]; in fact, the maximum size of a 3-incomplete set in G agrees with the maximum size of a sumfree set in G when G is cyclic.

Our methods are completely elementary, with Kneser's Theorem as the main tool. In Section 2 we review some standard terminology and notations and prove some auxiliary results, then in Section 3 we sketch the proof of Theorem 1 in the case when the order of the group is even.

2 Preliminaries

Here we present a few generic results that come useful in our proofs. We will use the following version of Kneser's Theorem.

Theorem 3 (Kneser's Theorem; [5])**.** *If $A_1, \ldots, A_h$ are nonempty subsets of a finite abelian group G, and H is the stabilizer subgroup of $A_1 + \cdots + A_h$ in G, then*

$$|A_1 + \cdots + A_h| \geq |A_1| + \cdots + |A_h| - (h-1)|H|.$$

Our first lemma is a simple application of Kneser's Theorem:

Lemma 4. *Suppose that G is a finite abelian group and that h is a positive integer. Let A be an h-incomplete subset of maximum size in G, and let H denote the stabilizer of hA in G. Then both A and hA are unions of full cosets of H; furthermore, if A and hA consist of k_1 and k_2 cosets of H, respectively, then*

$$k_2 \geq hk_1 - h + 1.$$

We will also use the following observation:

Lemma 5. *Suppose that G is a finite abelian group of order n and that h is a positive integer. Let H be a subgroup of G of index d for some $d \in \mathbb{N}$, and let ϕ be the canonical map from G to G/H. Suppose further that B is a subset of G/H, and set $A = \phi^{-1}(B)$. Then $|A| = \frac{n}{d} \cdot |B|$ and $|hA| = \frac{n}{d} \cdot |hB|$.*

Our next result takes advantage of the fact that the elements of a finite abelian group have a natural ordering. We review some background and introduce a useful result.

When G is cyclic and of order n, we identify it with $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. More generally, G has a unique *type* $(n_1, \ldots, n_r)$, where r and $n_1, \ldots, n_r$ are positive integers so that $n_1 \geq 2$, n_i is a divisor of n_{i+1} for $i = 1, \ldots, r-1$, and

$$G \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r};$$

here r is the *rank* of G and n_r is the *exponent* of G.

The above factorization of G allows us to arrange the elements in lexicographic order and then consider the 'first' m elements in G. Namely, suppose that m is a nonnegative integer less than n; we then have unique integers $q_1, \ldots, q_r$, so that $0 \leq q_k < n_k$ for each $1 \leq k \leq r$, and

$$m = \sum_{k=1}^{r} q_k n_{k+1} \cdots n_r.$$

For simplicity, we assume $q_r \geq 1$, in which case the first m elements in G range from the zero element to $(q_1, \ldots, q_{r-1}, q_r - 1)$ and thus form the set

$$\mathcal{I}(G, m) = \bigcup_{k=1}^{r} \{q_1\} \times \cdots \times \{q_{k-1}\} \times \{0, 1, \ldots, q_k - 1\} \times \mathbb{Z}_{n_{k+1}} \times \cdots \times \mathbb{Z}_{n_r}.$$

The advantage of considering these initial sets is that their h-fold sumsets are also initial sets. Indeed, assuming for simplicity that $hq_k < n_k$ for each k, we find that $h\mathcal{I}(G, m)$ consists of the elements from the zero element to $(hq_1, \ldots, hq_{r-1}, hq_r - h)$, and thus

$$h\mathcal{I}(G, m) = \mathcal{I}(G, hm - h + 1).$$

We will also employ a slight modification of $\mathcal{I}(G, m)$ where its last element is replaced by the next one in the lexicographic order. To avoid degenerate cases, we further assume that $q_r \geq 3$, in which case we have

$$\mathcal{I}^*(G, m) = \mathcal{I}(G, m-1) \cup \{(q_1, \ldots, q_{r-1}, q_r)\};$$

an easy calculation shows that

$$h\mathcal{I}^*(G, m) = \mathcal{I}(G, hm - 1) \cup \{(hq_1, \ldots, hq_{r-1}, hq_r)\}.$$

We can summarize these calculations, as follows.

Proposition 6. *Suppose that the finite abelian group G is of type $(n_1, \ldots, n_r)$. Let $0 \leq m < n$, and let $q_1, \ldots, q_r$ be the unique integers with $0 \leq q_k < n_k$ for each $1 \leq k \leq r$ for which*

$$m = \sum_{k=1}^{r} q_k n_{k+1} \cdots n_r.$$

Let h be a positive integer for which $hq_k < n_k$ for each $1 \leq k \leq r$. Then for the m-subsets $\mathcal{I}(G, m)$ and $\mathcal{I}^*(G, m)$ of G we have the following:

1. *If $q_r \geq 1$, then $|h\mathcal{I}(G, m)| = hm - h + 1$.*

2. *If $q_r \geq 3$, then $|h\mathcal{I}^*(G, m)| = hm$.*

3 Sketch of the proof for two-fold sumsets

In this section we outline the proof of Theorem 1 in the case when the order of the group is even: Theorem 9.

The critical number $\chi(G,2)$ is as follows.

Proposition 7. *For any abelian group G of order n we have*

$$\chi(G,2) = \lfloor n/2 \rfloor + 1.$$

We now turn to finding

$$S(G,2) = \{|2A| \ : \ A \subset G,\ |A| = \lfloor n/2 \rfloor\ ,\ 2A \neq G\}.$$

Our proof builds on the following result that may be of independent interest.

Theorem 8. *Let G be a finite abelian group of even order whose exponent is not divisible by 4, and suppose that A is a subset of G of size $|A| = n/2$. Then G has a subgroup H of order $n/2$ for which*

$$|A \cap H| \neq |A \cap (G \setminus H)|.$$

We note that the claim of Theorem 8 may be false in groups with exponent divisible by 4. For example, in $\mathbb{Z}_2 \times \mathbb{Z}_4$, the set $\mathbb{Z}_2 \times \{0,1\}$ intersects all three subgroups in two elements.

We are now ready to determine $S(G,2)$. Here we present the proof in the case when n is even.

Theorem 9. *If the exponent of G is divisible by 4, then*

$$S(G,2) = \{n - n/d \ : \ d|n,\ 2|d\}\,;$$

if the exponent of G is even but not divisible by 4, then

$$S(G,2) = \{n - n/d \ : \ d|n,\ 2|d,\ d \neq 4\}\,.$$

Proof: Using the notations of Lemma 4, we have $|A| = n/2 = k_1 n/d$ where d is the index of the stabilizer subgroup of $2A$. This implies that d is even and $k_1 = d/2$; using Lemma 4 again yields $k_2 \geq d-1$ and thus $|2A| = k_2 n/d$ equals n or $n - n/d$. Therefore, we have

$$S(G,2) \subseteq \{n - n/d \ : \ d|n,\ 2|d\}\,.$$

When the exponent of G is congruent to 2 mod 4, then we can rule out $d = 4$, as follows. By Theorem 8, G has a subgroup H of index 2 for which $H \cap A$ and $(G \setminus H) \cap A$ have different sizes; let $A = A_1 \cup A_2$ where A_1 and A_2 are subsets of different cosets of H. Without loss of generality, we assume that $|A_1| > n/4$, and thus $2A_1 = H$. If A_2 were to be empty, then A is a full coset of H, and thus $|2A| = n/2 \neq 3n/4$. Otherwise, $|A_1 + A_2| \geq |A_1| > n/4$, which implies that $|2A| \geq |2A_1| + |A_1 + A_2| > 3n/4$.

What remains is the proof that all remaining values arise as sumset sizes. This is clearly true when $d = 2$, or when $d = 4$ and the exponent of G is divisible by 4. Suppose now that d is an even divisor of n and $d > 4$. According to Lemma 5, it suffices to prove that every group K of order d contains some subset B of size $d/2$ for which $|2B| = d - 1$. Let H be any subgroup of index 2 in K, and set $B = (H \setminus \{h\}) \cup \{g\}$, where h and g are arbitrary elements of H and $K \setminus H$, respectively. Since $|H \setminus \{h\}| = d/2 - 1 > d/4$, we get $2(H \setminus \{h\}) = H$ and thus $2B = G \setminus \{h + g\}$. Therefore, $|2B| = d - 1$, and our proof is complete. $\square$

References

[1] B. Bajnok. "The h-critical number of finite Abelian groups". In: *Unif. Distrib. Theory* 10.2 (2015), pp. 93–115.

[2] B. Bajnok. *Additive combinatorics*. Discrete Mathematics and its Applications (Boca Raton). A menu of research problems. CRC Press, Boca Raton, FL, 2018, pp. xix+390. DOI: 10.1201/9781351137621.

[3] P. H. Diananda and H. P. Yap. "Maximal sum-free sets of elements of finite groups". In: *Proc. Japan Acad.* 45 (1969), pp. 1–5.

[4] B. Green and I. Z. Ruzsa. "Sum-free sets in abelian groups". In: *Israel J. Math.* 147 (2005), pp. 157–188. DOI: 10.1007/BF02785363.

[5] M. Kneser. "Abschätzung der asymptotischen Dichte von Summenmengen". In: *Math. Z.* 58 (1953), pp. 459–484. DOI: 10.1007/BF01174162.

[6] V. F. Lev. "Stability result for sets with $3A \neq \mathbb{Z}_5^n$". In: *J. Combin. Theory Ser. A* 157 (2018), pp. 334–348. DOI: 10.1016/j.jcta.2018.03.008.

On the Heesch number in $\mathbb{E}^d$

Bojan Bašić[1], Anna Slivková[2]

Department of Mathematics and Informatics, University of Novi Sad, Serbia

The full version of this work will be published elsewhere. The authors acknowledge financial support of the Ministry of Education, Science and Technological Development of the Republic of Serbia (Grant No. 451-03-9/2021-14/200125).

Abstract

In a recent article, it was shown that the Heesch number in $\mathbb{E}^d$ is asymptotically unbounded for $d \to \infty$, by showing that, for each d of the form 2^k, there exists a hypersolid in $\mathbb{E}^d$ whose Heesch number equals $d-1$. We here show that the same holds not only for d of the form 2^k, but for any d, $d \geqslant 2$.

1 Introduction

Within combinatorial geometry, problems on *tessellations* (or *tilings*) occupy one of the central spots. A tessellation of the Euclidean plane $\mathbb{E}^2$ is defined as a set $\mathcal{T}$ consisting of closed topological discs, where the elements of $\mathcal{T}$ have pairwise disjoint interiors and $\bigcup \mathcal{T} = \mathbb{E}^2$. The elements of $\mathcal{T}$ are called *tiles*. The monograph [4] is taken for a very thorough compendium of various problems of tilings, as well as the theoretical background.

The *Heesch number* of a figure (introduced in [5]) represents a kind of measure that expresses, loosely speaking, how "far" we can advance toward a tiling of the whole plane using the given figure (the greater Heesch number is, we can advance "further"; and the Heesch number is infinite if and only if the plane can be tiled by congruent copies of the given figure). In an intuitive sense (it will be formally defined in the following section), the Heesch number counts the number of times the given figure can be completely surrounded by its congruent copies.

The question whether the set of all possible finite Heesch numbers is bounded from above is known as *Heesch's problem.* For almost full 20 years, the "record-holder" (in the Euclidean plane) had been a figure whose Heesch number is 5 [6], which was finally surpassed last year, when a figure whose Heesch number is 6 has been discovered [2]. Some different versions of the problem, such as the problem posed in the hyperbolic plane, and the problem posed for sets of more figures (at least three), have been solved ([7], respectively [1]); in both these cases, it turns out that the upper bound does not exist, that is, it is possible to construct a figure (respectively a set of figures) whose Heesch number is as large as we please.

Until recently, all the research on the Heesch number has been done pretty much exclusively within the two-dimensional space (that is, the plane). In [3], d-dimensional Heesch's problem is solved in the asymptotic sense. Namely, it is shown that, if we let $d \to \infty$, then there is no uniform upper bound on the set of all possible finite Heesch numbers in the space $\mathbb{E}^d$; in other words, given any nonnegative integer n, we can find a dimension d (depending on n) in which there exists a hypersolid whose Heesch number is finite and greater than n. In particular, it is shown that, for each d of the form 2^k, there exists a hypersolid in $\mathbb{E}^d$ whose Heesch number equals $d-1$. We here show that the same holds not only for d of the form 2^k, but for any d, $d \geqslant 2$.

[1]Email: bojan.basic@dmi.uns.ac.rs
[2]Email: anna.slivkova@dmi.uns.ac.rs

2 Definitions and results

We first give a formal definition of the Heesch number.

Definition 1. We say that a hypersolid C (a topological d-ball) in $\mathbb{E}^d$ can be *surrounded n times* if and only if there exist finite collections $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ of isometric copies of C such that:

- every two different hypersolids from $\{C\} \cup \bigcup_{i=1}^{n} \mathcal{C}_i$ have disjoint interiors;
- for each i, $1 \leqslant i \leqslant n$, each hypersolid from $\mathcal{C}_i$ has a common boundary point with some hypersolid from $\mathcal{C}_{i-1}$ (where by convention, we let $\mathcal{C}_0 = \{C\}$);
- for each i, $1 \leqslant i \leqslant n$, $\bigcup \left(\bigcup_{j=0}^{i} \mathcal{C}_j \right)$ is a closed topological d-ball such that $\bigcup \left(\bigcup_{j=0}^{i-1} \mathcal{C}_j \right)$ is completely contained in its interior.

The collection $\mathcal{C}_i$ is called the i^{th} *corona*.

Definition 2. The *Heesch number* of a given hypersolid C (a topological d-ball) in $\mathbb{E}^d$ is the maximal nonnegative integer n such that C can be surrounded n times. If such a maximum does not exist, then we define the Heesch number to be infinite.

We shall now define a hypersolid in $\mathbb{E}^d$ whose Heesch number is $d-1$. This will be the same hypersolid defined in [3], for which it was shown there that, if $d = 2^k$, then its Heesch number equals $d-1$. For the reader's convenience, we repeat the definition here. We start from a unit hypercube, and mark some of its facets (which are $(d-1)$-dimensional unit hypercubes) by "bumps" and "nicks" (arranged in a particular way that will be described in a moment), where each bump matches each nick. In particular, each bump or nick can be taken to be a right hypercone whose base is an $(n-1)$-dimensional (small) hyperball placed in the center of a facet of the considered hypercube, and whose axis is orthogonal to the facet; we call *bumps*, respectively *nicks*, such hypercones erected outwards, respectively inwards (with respect to the interior of the considered hypercube).

Definition 3. A *basic hypercube* is a hypersolid obtained in the described way that has d facets with bumps and $d-1$ facets with nicks (and 1 facet not marked by either), where, additionally, all d facets with bumps have a common vertex.

It easily follows from the definition that any two basic hypercubes are isometric. Figure 1 presents a picture of a 3-dimensional basic hypercube and a 4-dimensional basic hypercube.

The following lemma is also taken from [3] (note that its proof there does not rely on the fact that $d = 2^k$, and thus the lemma is valid in any space $\mathbb{E}^d$).

Lemma 4. *The Heesch number of a basic hypercube is at most $d-1$.*

Therefore, we are left to show the other inequality, that is, to show how to surround a basic hypercube $d-1$ times by its isometric copies. The constraint $d = 2^k$ has an essential role in the construction from [3] and it does not seem possible to make some amendments to the idea from there that would eliminate this constraint. That is why we here devise a completely different approach, that is not dependent on the form of d.

We introduce two more types of marked hypercubes, called *neutral δ-hypercubes* and *spikey δ-hypercubes*, where $\delta \in \{1, 2, \ldots, d\}$. Each of them is a δ-dimensional unit hypercube, with facets marked by bumps and nicks as follows. A *neutral δ-hypercube* has δ facets marked by bumps and the other δ facets marked by nicks, where all δ facets with bumps have a common vertex (and, clearly, the same holds for the δ facets with nicks). A *spikey δ-hypercube* can be obtained from a neutral δ-hypercube by replacing one nick by a bump (in other words, a spikey δ-hypercube has $\delta+1$ facets marked by bumps and $\delta-1$ facets marked by nicks, where all $\delta-1$ facets with nicks have two common vertices).

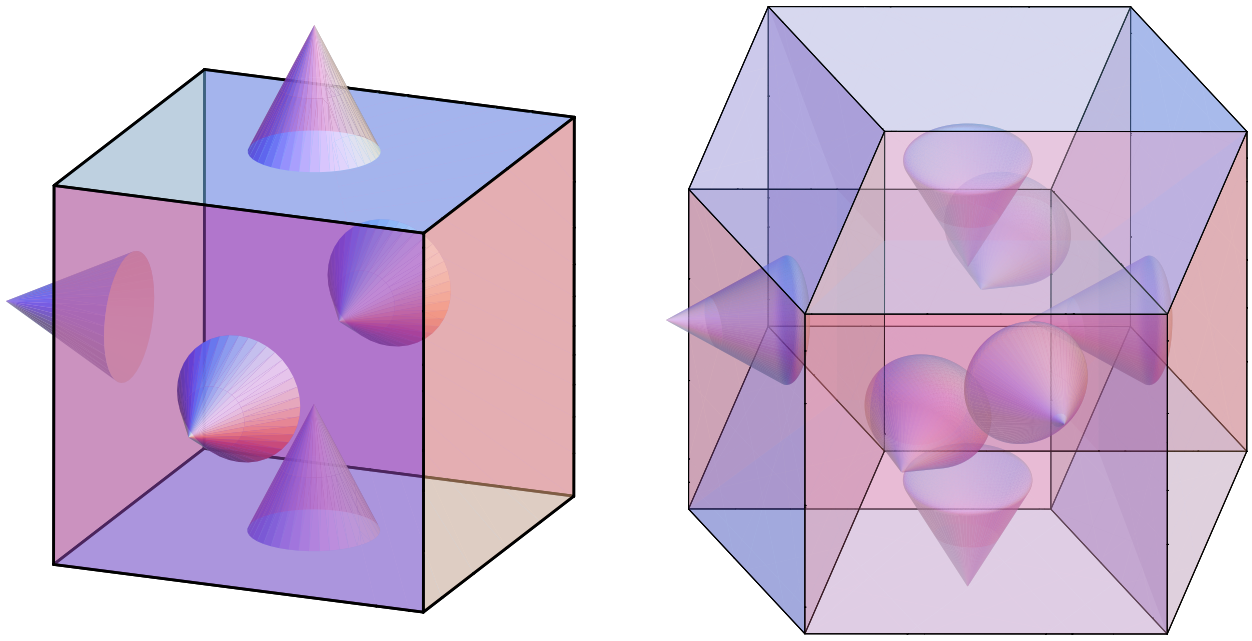


Figure 1: A 3-dimensional and a 4-dimensional basic hypercube.

We shall show that a hypercube of side $2d$ in $\mathbb{E}^d$ can be arranged (up to bumps and nicks) from $(2d)^d$ basic hypercubes (this is actually more than we need; for our purpose, it would be enough to obtain a hypercube of side $2d-1$). Of course, in such an arrangement, every two basic hypercubes that have a common facet must have the corresponding facets marked in a matching way.

We define the following auxiliary structure.

Definition 5. Let $\delta \in \{1, 2, \ldots, d\}$ and $i \in \{0, 1, \ldots, d-1\}$. A (δ, i)-*layer* is a δ-dimensional hypercube of side d, up to bumps and nicks, that satisfies the following two properties:

- it can be obtained by arranging a total of $(d-\delta)d^{\delta-1}$ neutral δ-hypercubes and $\delta d^{\delta-1}$ spikey hypercubes;
- the structure can be placed in the integer grid, with centers of the constituting hypercubes at the coordinates $\{0, 1, \ldots, d-1\}^\delta$, in such a way that the constituting neutral δ-hypercubes are centered exactly at the coordinates

$$\left\{(c_1, c_2, \ldots, c_\delta) : \sum_{k=1}^{\delta} c_k \in \{i, i+1, i+2, \ldots, i+(d-\delta)-1\}\right\}$$

  (the operations are performed modulo d).

The introduced layers can be used to inductively (by induction on δ) prove the following lemma.

Lemma 6. *A (δ, i)-layer exists for each δ and i, with $\delta \in \{1, 2, \ldots, d\}$ and $i \in \{0, 1, \ldots, d-1\}$.*

The following lemma establishes a link between basic hypercubes and layers.

Lemma 7. *For each i, $i \in \{0, 1, \ldots, d-1\}$, a total of $(2d)^d$ basic hypercubes can be arranged to form a structure that is, when bumps and nicks are ignored, the same as a (d, i)-layer scaled by factor 2.*

Finally, we have our main statement.

Theorem 8. *For any $d \in \mathbb{N}$, $d \geqslant 2$, the Heesch number of a basic hypercube in d dimensions equals $d-1$.*

Proof. By Lemma 4 we have that $d-1$ is an upper bound on the Heesch number of a d-dimensional basic hypercube. We shall see that $d-1$ is also a lower bound, for which we use Lemma 7. Namely, note that, for any i, a (d,i)-layer scaled by factor 2 is a hypercube of side $2d$; therefore, $(2d)^d$ basic hypercubes can be arranged to form such a hypercube (up to bumps and nicks). Then, trivially, $(2d-1)^d$ basic hypercubes can be arranged to form a hypercube of side $2d-1$, which means that a basic hypercube can be surrounded $d-1$ times by its isometric copies. Therefore, $d-1$ is also a lower bound on the Heesch number of a d-dimensional basic hypercube, which was to be proved. □

References

[1] B. Bašić. "The Heesch number for multiple prototiles is unbounded". In: *C. R. Math. Acad. Sci. Paris* 353.8 (2015), pp. 665–669. DOI: 10.1016/j.crma.2015.05.002.

[2] B. Bašić. "A figure with Heesch number 6: pushing a two-decade-old boundary". In: *Math. Intelligencer* 43.3 (2021), pp. 50–53. DOI: 10.1007/s00283-020-10034-w.

[3] B. Bašić and A. Slivková. "Asymptotical unboundedness of the Heesch number in $\mathbb{E}^d$ for $d \to \infty$". In: *Discrete Comput. Geom.* 67.1 (2022), pp. 328–337. DOI: 10.1007/s00454-020-00254-4.

[4] B. Grünbaum and G. C. Shephard. *Tilings and patterns*. A Series of Books in the Mathematical Sciences. An introduction. W. H. Freeman and Company, New York, 1989, pp. xii+446.

[5] H. Heesch. *Reguläres Parkettierungsproblem*. Arbeitsgemeinschaft für Forschung des Landes Nordrhein-Westfalen, Heft 172. Westdeutscher Verlag, Cologne, 1968, p. 96.

[6] C. E. Mann. "On Heesch's problem and other tiling problems". PhD thesis. University of Arkansas, 2001.

[7] A. S. Tarasov. "On the Heesch number for the Lobachevskiĭ plane". In: *Mat. Zametki* 88.1 (2010), pp. 97–102. DOI: 10.1134/S0001434610070096.

2-LC triangulated manifolds are exponentially many

Bruno Benedetti[1], Marta Pavelka[2]

Dept. of Mathematics, University of Miami

Abstract

We introduce "t-LC triangulated manifolds" as those triangulations obtainable from a tree of d-simplices by recursively identifying two boundary $(d-1)$-faces whose intersection has dimension at least $d-t-1$. The t-LC notion interpolates between the class of LC manifolds introduced by Durhuus-Jonsson (corresponding to the case $t=1$), and the class of all manifolds (case $t=d$). Benedetti–Ziegler proved that there are at most $2^{d^2\,N}$ triangulated 1-LC d-manifolds with N facets. Here we prove that there are at most $2^{\frac{d^3}{2}N}$ triangulated 2-LC d-manifolds with N facets. This extends an intuition by Mogami for $d=3$ to all dimensions.

We also introduce "t-constructible complexes", interpolating between constructible complexes (the case $t=1$) and all complexes (case $t=d$). We show that all t-constructible pseudomanifolds are t-LC, and that all t-constructible complexes have (homotopical) depth larger than $d-t$. This extends the famous result by Hochster that constructible complexes are (homotopy) Cohen–Macaulay.

Details, proofs, and more can be found in our preprint `https://arxiv.org/pdf/2106.12136.pdf`.

1 Introduction

Since the Sixties Tullio Regge [12, 13, 14] and many other physicists and mathematicians, cf. e.g. [1], [2], [10], have worked to develop a discrete version of quantum gravity. In Weingarten's dynamical triangulations (or "DT") setup [15], smooth manifolds are approximated by equilateral triangulations. This allows to translate all metric aspects, such as curvature and volume, into simpler combinatorial calculations; for example, the partition function for gravity, which is a path integral over all possible metrics, becomes an infinite sum over all triangulations. The downside of this powerful simplification method is a convergence issue. For example, the partition function diverges to infinity, unless one restricts the sum to triangulations into a certain class, and such class happens to have exponential size. In fact, for any fixed $d \geq 2$, there are more than exponentially many triangulated d-manifolds with N facets. Here two triangulations are considered equal if they are 'combinatorially isomorphic': That is, if up to relabeling the vertices they have the same face poset.

In an important step for this program, Durhuus and Jonsson [7] defined "locally constructible" (LC) manifolds as those triangulated manifolds obtainable from a tree of d-simplices by recursively identifying two boundary facets whose intersection has *codimension one*. They proved that LC 3-manifolds are exponentially many [7]; and also in higher dimensions, LC d-manifolds are less than $2^{d^2\,N}$ [6]. Since all polytope boundaries are LC, this idea lead to a first proof that polytopes with N facets, in fixed dimension, are exponentially many [6].

[1]Email: bruno@math.miami.edu. Supported by NSF Grant 1855165, "Geometric Combinatorics and Discrete Morse Theory".

[2]Email: pavelka@math.miami.edu

Here we define 2-*LC manifolds* as those obtainable from a tree of d-simplices by recursively identifying two boundary facets that intersect in *codimension one or two*. We prove that this broader class has also exponential size:

Theorem 1. *For fixed $d \geq 3$, the number of combinatorially distinct 2-LC d-manifolds with N facets is smaller than $2^{\frac{d^3}{2}N}$.*

The bound can be extended also to 2-LC *quasi-manifolds*, which are pseudomanifolds with particularly nice face links, but not to arbitrary 2-LC pseudomanifolds, which are more than exponentially many. Theorem 1 gives a precise mathematical formulation and extends to all dimensions an intuition by Mogami [11], who worked on 2-LC 3-spheres. The crucial ingredient for this novel exponential upper bound is the planarity of the links of all $(d-3)$-faces. In general, the link of a k-dimensional face in a triangulated d-dimensional manifold (without boundary) is a homology-sphere of dimension $(d-k-1)$. However, since homology-spheres that are not spheres exist only in dimension 3 and higher, when $k = d-3$ all links of $(d-3)$-faces are indeed homeomorphic to S^2.

This brings topology into the picture. Durhuus and Jonsson conjectured in 1995 that all 3-spheres and 3-balls are LC [7]. The conjecture was disproved in 2011 by the first author and Ziegler [6]. The weaker conjecture by Mogami [11] that all 3-balls are 2-LC was also recently disproved by the first author [4]. Thus there is little hope that these combinatorial cutoffs may encompass entire topologies. But there are two other reasons why the LC notion is of mathematical importance, beyond the enumerative aspect mentioned above:

(a) All LC-triangulable manifolds are simply-connected, and conversely, all simply-connected PL manifolds of dimension $\neq 4$ admit an LC subdivision [3];

(b) All shellable and all constructible manifolds are LC [6].

Both results above are still valid if one replaces "LC" with "2-LC". This triggers a natural curiosity, namely, whether for the result (b) above, for the 2-LC case, one could say more. Perhaps the 'constructible' assumption can be weakened?

To answer this curiosity, we define more generally "t-LC triangulated manifolds" as those obtainable from a tree of d-simplices by recursively identifying two boundary $(d-1)$-faces whose intersection has dimension at least $d-t-1$. This notion interpolates between LC manifolds (which are the same as 1-LC) and all manifolds (the same as d-LC); the case $t = d-1$ was also previously studied [4]. In parallel, we introduce "t-constructible complexes" as a generalization of constructible complexes, which correspond to the $t = 1$ case. Intuitively, t-constructible d-complexes are defined recursively as those obtained by gluing two t-constructible d-complexes at a codimension-one subcomplex whose $(d-t)$-skeleton is constructible.

With these two new properties, we prove the following generalization of the well-known result by Hochster [9] that all constructible d-complexes are Cohen–Macaulay:

Theorem 2. *All t-constructible d-complexes have homotopical depth larger than $d-t$. Moreover, all t-constructible pseudomanifolds are t-LC.*

The converse of Theorem 2 is false, even if we restrict ourselves to 3-manifolds. In fact, in [5] there are two explicit examples (with 13 and 16 vertices, respectively) of two 3-spheres containing a non-trivial knot that is realized by just three edges in their 1-skeleton; the knots are the trefoil and the square knot, respectively. These examples have homotopical depth 3 because they are spheres, and are 1-LC by computation [5], but they are not 1-constructible because of the knot [8].

References

[1] J. Ambjørn, B. Durhuus, and T. Jonsson. *Quantum geometry*. Cambridge Monographs on Mathematical Physics. A statistical field theory approach. Cambridge University Press, Cambridge, 1997, pp. xiv+363. DOI: 10.1017/CBO9780511524417.

[2] J. Ambjørn, J. Jurkiewicz, and R. Loll. "Deriving spacetime from first principles". In: *Ann. Phys.* 19.3-5 (2010), pp. 186–195. DOI: 10.1002/andp.201010414.

[3] B. Benedetti. "Smoothing discrete Morse theory". In: *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 16.2 (2016), pp. 335–368.

[4] B. Benedetti. "Mogami manifolds, nuclei, and $3D$ simplicial gravity". In: *Nuclear Phys. B* 919 (2017), pp. 541–559. DOI: 10.1016/j.nuclphysb.2017.04.001.

[5] B. Benedetti and F. H. Lutz. "Knots in collapsible and non-collapsible balls". In: *Electron. J. Combin.* 20.3 (2013), Paper 31, 29.

[6] B. Benedetti and G. M. Ziegler. "On locally constructible spheres and balls". In: *Acta Math.* 206.2 (2011), pp. 205–243. DOI: 10.1007/s11511-011-0062-2.

[7] B. Durhuus and T. Jónsson. "Remarks on the entropy of 3-manifolds". In: *Nuclear Phys. B* 445.1 (1995), pp. 182–192. DOI: 10.1016/0550-3213(95)00207-9.

[8] M. Hachimori and G. M. Ziegler. "Decompositons of simplicial balls and spheres with knots consisting of few edges". In: *Math. Z.* 235.1 (2000), pp. 159–171. DOI: 10.1007/s002090000129.

[9] M. Hochster. "Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes". In: *Ann. of Math. (2)* 96 (1972), pp. 318–337. DOI: 10.2307/1970791.

[10] R. Loll. "Quantum gravity from causal dynamical triangulations: a review". In: *Classical Quantum Gravity* 37.1 (2020), pp. 013002, 50. DOI: 10.1088/1361-6382/ab57c7.

[11] T. Mogami. "On the entropy bound of three-dimensional simplicial gravity". In: *Progr. Theoret. Phys.* 94.1 (1995), pp. 157–161. DOI: 10.1143/PTP.94.157.

[12] G. Ponzano and T. Regge. "Semiclassical limit of Racah coefficients". In: *Spectroscopic and Group Theoretical Methods in Physics*. Ed. by F. Bloch. North-Holland Publ. Co., Amsterdam, 1969, pp. 1–58.

[13] T. Regge. "General relativity without coordinates". In: *Nuovo Cimento (10)* 19 (1961), pp. 558–571.

[14] T. Regge and R. M. Williams. "Discrete structures in gravity". In: *J. Math. Phys.* 41.6 (2000), pp. 3964–3984. DOI: 10.1063/1.533333.

[15] D. Weingarten. "Euclidean quantum gravity on a lattice". In: *Nuclear Physics B* 210.2 (1982), pp. 229–245. DOI: https://doi.org/10.1016/0550-3213(82)90241-3.

Accessible set systems and a conjecture on Cohen-Macaulay binomial edge ideals

Davide Bolognini[1*], Antonio Macchia[2†], Giancarlo Rinaldo[3‡], Francesco Strazzanti[4§]

[*]Dip. di Ingegneria Industriale e Scienze Matematiche, Università Politecnica delle Marche, Via Brecce Bianche, 60131 Ancona, Italy
[†]Fach. Mathematik und Informatik, Freie Universität Berlin, Arnimallee 2, 14195 Berlin, Germany
[‡]Dip. di Matematica e Informatica, Fisica e Scienze della Terra, Università di Messina, Viale Ferdinando Stagno d'Alcontres 31, 98166 Messina, Italy
[§]Dip. di Matematica "Giuseppe Peano", Università degli Studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy

The full version of this work will appear in [2] and will be published elsewhere.

Abstract

Binomial edge ideals are generated by the 2-minors of a generic matrix with two rows whose column indices correspond to the edges of a graph. They naturally generalize the ideals of 2-minors and appear in Algebraic Statistics. Many algebraic properties of binomial edge ideals can be studied through the cut sets of the associated graph, which are special sets of vertices whose removal disconnects the graph. A recent conjecture by some of the authors states that Cohen-Macaulay binomial edge ideals are exactly the binomial edge ideals J_G whose graph is *accessible*, i.e., J_G is unmixed and the collection of cut sets of G is an accessible set system. Exploiting the structure of accessible graphs, we provide further evidence in support of this conjecture, proving that it holds for all graphs with up 15 vertices. The computations for graphs with $13, 14$ and 15 vertices could not be completed with a naive brute force approach.

1 Introduction

Binomial edge ideals are quadratic binomial ideals generated by the 2-minors of a matrix of variables with two rows whose columns are indexed by the edges of a finite simple graph. More precisely, given a finite simple graph G with n vertices, the *binomial edge ideal* of G is the ideal

$$J_G = (x_i y_j - x_j y_i : \{i, j\} \in E(G)) \subset K[x_i, y_i : i \in [n]],$$

where $E(G)$ is the edge set of G and K is a field. They were introduced at the same time in [8] and [14] and naturally generalize the ideals of 2-minors of a matrix with two rows. Moreover, they arise in the study of conditional independence statements in Algebraic Statistics [8, Section 4] and are a subclass of the so-called *Cartwright-Sturmfels ideals* [5, Section 3].

Many researchers studied algebraic properties and computed invariants of binomial edge ideals exploiting the combinatorics of the underlying graph, see e.g., [1, 3, 4, 10, 12, 18].

An important and hard question that is still open concerns a purely combinatorial characterization of Cohen-Macaulay binomial edge ideals, that would in turn show the independence of Cohen-Macaulayness

[1]Email: davide.bolognini.cast@gmail.com.
[2]Email: macchia.antonello@gmail.com. Research of A. M. supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – project number 454595616.
[3]Email: giancarlo.rinaldo@unime.it
[4]Email: francesco.strazzanti@gmail.com.

of the field. The same question can be asked for monomial ideals, which are associated to simplicial complexes through the well-known Stanley-Reisner correspondence. However, Cohen-Macaulayness of monomial ideals depends on the field K and, hence, such a general combinatorial characterization cannot exist.

A fundamental concept that comes up in the study of binomial edge ideals is the notion of cut set of a graph. A vertex v of a graph G is called a *cut vertex* if $G \setminus \{v\}$ has more connected components than G. More in general, a set of vertices S of G is a *cut set* (or *cut-point set*) if either $S = \varnothing$ or $c_G(S \setminus \{s\}) < c_G(S)$ for every $s \in S$, where $c_G(S)$ denotes the number of connected components of the graph $G \setminus S$ obtained from G by removing the vertices of S. To the best of our knowledge, even if this notion of cut sets resembles other concepts in Graph Theory, it has not been studied yet.

The cut sets of a graph G are in bijection with the minimal prime ideals of J_G and one can translate several algebraic properties of J_G in terms of cut sets. For example, this can be done for unmixedness, which is weaker than Cohen-Macaulayness: J_G is *unmixed* (i.e., all its minimal primes have the same height) if and only if $c_G(S) = |S| + c$ for every $S \in \mathcal{C}(G)$, where $\mathcal{C}(G)$ is the collection of cut sets of G and c is the number of connected components of G, see [15, Lemma 2.5].

Several authors constructed classes of graphs whose binomial edge ideal is Cohen-Macaulay, see [3, 7, 9, 15, 16, 17]. In particular, in [3, Theorem 6.1] the first, second and fourth author proved that, if G is bipartite, the Cohen-Macaulayness of J_G is equivalent to the following condition on the cut sets of G:

$$\begin{aligned} &J_G \text{ is unmixed and } \mathcal{C}(G) \text{ is an } \textit{accessible set system}, \text{ i.e., for every non-empty } S \in \mathcal{C}(G) \\ &\text{there exists } s \in S \text{ such that } S \setminus \{s\} \in \mathcal{C}(G). \end{aligned} \tag{$*$}$$

Following [4], we call *accessible* a graph G that satisfies Conditions $(*)$. Notice that, since the unmixedness of J_G can also be expressed in terms of cut sets, the accessibility of G is a purely graph-theoretic condition.

In the follow-up paper [4], the same authors prove that for a general graph G,

$$J_G \text{ Cohen-Macaulay } \Rightarrow G \text{ accessible},$$

and show that to prove the reverse implication, it is enough to find a cut vertex v such that $J_{G\setminus\{v\}}$ is unmixed for all accessible graphs G. Moreover, they prove that this is the case when G is an accessible graph that is chordal or traceable. They state the following

Conjecture 1 ([4, Conjecture 1.1])**.** Let G be a graph. Then J_G is Cohen-Macaulay if and only if G is accessible. In particular, the Cohen-Macaulayness of J_G does not depend on the field K.

A first simplification in order to computationally check the conjecture for all graphs with a given number of vertices is that we can limit our search to connected graphs, see [4, Remark 2.4]. In [10], the third author and others verified Conjecture 1 for all connected graphs with up to 12 vertices.

Moreover, recently, in [19, Theorem 1.2] Saha and Sengupta showed that it is enough to prove Conjecture 1 for *blocks with whiskers*, which are graphs without cut vertices (*blocks*) with the addition of a pendant edge to some vertices of the block. In [2] we find an independent proof of this result.

The next computational step to verify Conjecture 1 or to find a counterexample consists in checking all blocks with whiskers with 13 vertices. However, in this case it is not possible to use a brute force search because the number of connected graphs with 13 vertices is $\sim 5 \cdot 10^{13}$. Hence, we need to develop some other strategies to make this computation doable.

In this talk I will present several theoretical results that allowed us to computationally check Conjecture 1 for all graphs with $13, 14$ and 15 vertices. The computations for this project have been performed on the server of the Laboratory of Cryptography, Department of Mathematics, University of Trento.

The next case, i.e., checking all blocks with whiskers with 16 vertices could still be doable with our current implementation but it would take several weeks to finish on a high performance computing cluster.

2 Computational results

Suppose that we want to generate all connected graphs on n vertices consisting of a block with exactly k whiskers attached to it, for some $k \geq 1$. To do that, first we consider all connected blocks B on $n-k$ vertices. We will then attach k whiskers to k distinct vertices $v_1, \ldots, v_k$ of B. We call $\overline{B}$ the graph obtained by adding $v_i w_i$ to B for $i = 1, \ldots, k$, where $w_1, \ldots, w_k$ are new vertices.

We now list all simplifications that we employed in our search and we refer to [4, 2] for their proofs. Some of these simplifications allow to filter blocks before adding whiskers.

1. We can exclude blocks B with vertices of degree 2.
2. We can exclude blocks B containing at least one free vertex w, i.e., a vertex that belongs to exactly one clique of B.
3. We can exclude a block with whiskers $\overline{B}$ if one of its cut vertices $v \in \{v_1, \ldots, v_k\}$ has degree $\deg_{\overline{B}}(v) > \frac{|V(B)|+k}{2} - 1$.
4. By [4, Proposition 4.10], we can exclude the graphs $\overline{B}$ such that the induced subgraph $\overline{B}[\{v_1, \ldots, v_k\}]$ is not connected.
5. By [4, Proposition 6.6], we can exclude the graphs $\overline{B}$ in which the induced subgraph $\overline{B}[\{v_1, \ldots, v_k\}]$ is a complete graph. This excludes the blocks with $k = 1, 2$ whiskers. Moreover, we can also exclude blocks with $k = 3$ whiskers by [19, Proposition 5.3] (in [2] we found an independent and simpler proof).
6. By [4, Proposition 4.11], we can exclude the graphs $\overline{B}$ in which some vertex of B that is not a cut vertex is not adjacent to some of $v_1, \ldots, v_k$.

Using these filters, we verified Conjecture 1 for all graphs with up to 15 vertices.

We implemented our algorithm in *C++* and *Python* and generated all blocks with a certain number of vertices with *Nauty* [13]. We highlight that the computation of the blocks with whiskers with 15 vertices took several days on a node with 4 CPU Xeon-Gold 5118 using at most 12 cores. For this purpose we parallelized the execution of the tasks with the tool *Parallel* [20]. An exhaustive search on all connected graphs with 12 vertices took about a month on the same machine [10]. In our current implementation the bottleneck is the generation of blocks with whiskers that satisfy the filters described above. This step is implemented in Python using the library *igraph* [6].

In Tables 1 and 2 we collect the number of accessible graphs with $n \leq 15$ vertices. Our contribution is in the last two columns of Table 1 and in Table 2. For graphs with $n \leq 12$ vertices, Lerda, Mascia, Rinaldo and Romeo in [10, 11] computed all *indecomposable* accessible graphs without applying any of the above simplifications, see the third column of Table 1 (we refer to [10] for the definition of indecomposable graph). We compute all accessible blocks with whiskers (see the fourth column of Table 1), most of which are indecomposable. Finally, in the last column of Table 1 we collect the number of accessible blocks with whiskers that satisfy all the filters described above. These are also indecomposable and one can see that the number of filtered accessible blocks with whiskers is very low compared to the numbers in the third and fourth columns.

In Table 2 we count the blocks with whiskers satisfying all the above filters, and the ones that are unmixed and accessible, subdivided according to the number of whiskers. Notice that in the last row of the table we do not have the number of blocks with 11 vertices and 4 whiskers because in this case our implementation keeps only the graphs whose binomial edge ideal is unmixed.

Number of vertices	Connected graphs	Accessible indecomposable graphs [10, 11]	Accessible blocks with whiskers [2]	Filtered accessible blocks with whiskers [2]
2	1	1	0	0
3	2	1	1	0
4	6	1	2	0
5	21	2	2	0
6	112	5	4	0
7	853	15	10	0
8	11,117	51	32	0
9	261,080	194	109	0
10	11,716,571	833	467	1
11	1,006,700,565	3824	2051	1
12	164,059,830,476	19343	10550	8

Table 1: Accessible graphs

Number of vertices ↓	Blocks with whiskers				Unmixed blocks with whiskers				Accessible blocks with whiskers				
(Number of whiskers) →	4	5	6	7	4	5	6	7	4	5	6	7	Tot.
8	0	-	-	-	0	-	-	-	0	-	-	-	0
9	0	-	-	-	0	-	-	-	0	-	-	-	0
10	8	0	-	-	1	0	-	-	1	0	-	-	1
11	26	3	-	-	0	1	-	-	0	1	-	-	1
12	7,688	336	8	-	8	6	0	-	3	5	0	-	8
13	125,474	7,297	111	-	0	8	2	-	0	3	2	-	5
14	49,256,463	1,676,045	18,741	73	31	144	72	0	2	65	55	0	122
15	?	89,636,101	1,167,017	5,310	7	156	146	9	0	56	90	9	155

Table 2: Filtered accessible blocks with whiskers

Acknowledgement. *The computations for this project have been performed on the server of the Laboratory of Cryptography, Department of Mathematics, University of Trento.*

References

[1] A. Banerjee and L. Núñez-Betancourt. "Graph connectivity and binomial edge ideals". In: *Proc. Amer. Math. Soc.* 145.2 (2017), pp. 487–499. DOI: 10.1090/proc/13241.

[2] D. Bolognini, A. Macchia, G. Rinaldo, and F. Strazzanti. *Towards a characterization of Cohen-Macaulay bino- mial edge ideals*. In preparation.

[3] D. Bolognini, A. Macchia, and F. Strazzanti. "Binomial edge ideals of bipartite graphs". In: *European J. Combin.* 70 (2018), pp. 1–25. DOI: 10.1016/j.ejc.2017.11.004.

[4] D. Bolognini, A. Macchia, and F. Strazzanti. "Cohen–Macaulay binomial edge ideals and accessible graphs". In: *J. Algebraic Combin.* 55.4 (2022), pp. 1139–1170. DOI: 10.1007/s10801-021-01088-w.

[5] A. Conca, E. De Negri, and E. Gorla. "Cartwright-Sturmfels ideals associated to graphs and linear spaces". In: *J. Comb. Algebra* 2.3 (2018), pp. 231–257. DOI: 10.4171/JCA/2-3-2.

[6] G. Csárdi and T. Nepusz. "The igraph software package for complex network research". In: *Inter. J. Comp. Syst* 1695 (2006). https://igraph.org, pp. 1–9.

[7] V. Ene, J. Herzog, and T. Hibi. "Cohen-Macaulay binomial edge ideals". In: *Nagoya Math. J.* 204 (2011), pp. 57–68. DOI: 10.1215/00277630-1431831.

[8] J. Herzog, T. Hibi, F. Hreinsdóttir, T. Kahle, and J. Rauh. "Binomial edge ideals and conditional independence statements". In: *Adv. in Appl. Math.* 45.3 (2010), pp. 317–333. DOI: 10.1016/j.aam.2010.01.003.

[9] D. Kiani and S. Saeedi Madani. "Some Cohen-Macaulay and unmixed binomial edge ideals". In: *Comm. Algebra* 43.12 (2015), pp. 5434–5453. DOI: 10.1080/00927872.2014.952734.

[10] A. Lerda, C. Mascia, G. Rinaldo, and F. Romeo. *(S_2)-condition and Cohen-Macaulay binomial edge ideals*. Preprint. 2021. DOI: 10.48550/ARXIV.2107.04539.

[11] A. Lerda, C. Mascia, G. Rinaldo, and F. Romeo. *The repository of Cohen-Macaulay binomial edge ideals of graphs with $n \leq 12$ vertices*. http://www.giancarlorinaldo.it/s2binomials. 2021.

[12] K. Matsuda and S. Murai. "Regularity bounds for binomial edge ideals". In: *J. Commut. Algebra* 5.1 (2013), pp. 141–149. DOI: 10.1216/jca-2013-5-1-141.

[13] B. D. McKay and A. Piperno. "Practical graph isomorphism, II". In: *J. Symbolic Comput.* 60 (2014). The software package *Nauty* is available at http://cs.anu.edu.au/~bdm/nauty/, pp. 94–112. DOI: 10.1016/j.jsc.2013.09.003.

[14] M. Ohtani. "Graphs and ideals generated by some 2-minors". In: *Comm. Algebra* 39.3 (2011), pp. 905–917. DOI: 10.1080/00927870903527584.

[15] A. Rauf and G. Rinaldo. "Construction of Cohen-Macaulay binomial edge ideals". In: *Comm. Algebra* 42.1 (2014), pp. 238–252. DOI: 10.1080/00927872.2012.709569.

[16] G. Rinaldo. "Cohen-Macauley binomial edge ideals of small deviation". In: *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* 56(104).4 (2013), pp. 497–503.

[17] G. Rinaldo. "Cohen-Macaulay binomial edge ideals of cactus graphs". In: *J. Algebra Appl.* 18.4 (2019), p. 1950072. DOI: 10.1142/S0219498819500725.

[18] M. Rouzbahani Malayeri, S. Saeedi Madani, and D. Kiani. "A proof for a conjecture on the regularity of binomial edge ideals". In: *J. Combin. Theory Ser. A* 180 (2021), Paper No. 105432, 9. DOI: 10.1016/j.jcta.2021.105432.

[19] K. Saha and I. Sengupta. *Cohen-Macaulay Binomial edge ideals in terms of blocks with whiskers*. Preprint. 2022. DOI: 10.48550/ARXIV.2203.04652.

[20] O. Tange. "GNU Parallel - The Command-Line Power Tool". In: *;login: The USENIX Magazine* 36 (1 2011), pp. 42–47.

Logarithmic convergence of projective planes

Márton Borbényi[1*], Panna Fekete[2*,†], Aranka Hrušková[3†,‡], Ander Lamaison[4§]

[*]Eötvös Loránd University, Budapest, Hungary
[†]Alfréd Rényi Institute of Mathematics, Budapest, Hungary
[‡]Central European University, Budapest, Hungary
[§]Masaryk University, Brno, Czech Republic

Abstract

In this paper, we study the so-called log-convergence of graphs defined by Szegedy. We answer positively his question of whether the sequence $(G_q)_q^\infty$ of the incidence graphs of finite projective planes log-converges and whether the limit coincides with that of a particular random graph model. Moreover, we show that the sequence is still convergent if q ranges not just over primes, but over prime powers.

1 Introduction

Since its introduction in the 1980s, the theory of graph limits has been applied to numerous problems in extremal graph theory, and it has been generalized and extended to many different discrete structures. A particular focus has been placed on dense graph limits: limits of graph sequences in which the edge density is bounded away from 0. While there has been some success in defining a limit theory in other particular classes, such as graphs with bounded degree, there is no universally agreed upon approach toward limits and convergence in the entire sparse regime.

One such approach was laid out by Szegedy [3], namely logarithmic convergence. This model of convergence was motivated by Sidorenko's conjecture [2], in which the central inequality becomes linear after taking logarithms. Let $t(H, G)$ denote the probability that a random map from $V(H)$ to $V(G)$ is a graph homomorphism (i.e., it maps edges to edges).

Definition 1. Let $(G_k)_{k=1}^\infty$ be a graph sequence. We say that the sequence is *log-convergent* if, for any pair of bipartite graphs H_1, H_2 with at least one edge, the ratio

$$\frac{\log t(H_1, G_n)}{\log t(H_2, G_n)}$$

has a limit.

Since we have $\frac{\log(|V(G_n)|^{|V(H_1)|})}{\log(|V(G_n)|^{|V(H_2)|})} = \frac{|V(H_1)|}{|V(H_2)|}$ is a constant, we can replace the parameter $t(H, G)$ in the definition of log-convergence by $|\mathsf{hom}(H, G)|$, the number of homomorphisms from H to G.

[1]Email: marton.borbenyi@gmail.com. Supported by the ÚNKP-21-2 New National Excellence Program of the Ministry for Innovation and Technology from the source of the National Research, Development and Innovation Fund.

[2]Email: fekete.panna.timea@renyi.hu. Prepared with the professional support of the Doctoral Student Scholarship Program of the Co-operative Doctoral Program of the Ministry of Innovation and Technology financed from the National Research, Development and Innovation Fund.

[3]Email: umim.cist@gmail.com

[4]Email: lamaison@fi.muni.cz. This research was supported by the MUNI Award in Science and Humanities of the Grant Agency of Masaryk University.

Szegedy studied the convergence of the random bipartite graph model. For the purpose of our paper, we will only introduce one particular case of this model, and using a parametrization that is very different from that used by Szegedy. In our model, which depends on a parameter q, the two vertex sets X and Y have size q^2, and any two vertices in different parts are joined with an edge independently with probability q^{-1}. We denote the resulting graph by R_q.

Szegedy proved that, with probability 1, the sequence $(R_q)_q^\infty$ is log-convergent, and described the limit explicitly. We will describe this limit using a slightly different language. We say that a bipartite graph H' is a collapse of H if there is a homomorphism from H to H' that is surjective on both vertices and edges. We denote the set of all collapses of H by $\mathcal{C}(H)$. For a bipartite graph H, define

$$\hat{i}_R(H) = 2|V(H)| - |E(H)|, \qquad \hat{h}_R(H) = \max_{H' \in \mathcal{C}(H)} \hat{i}_R(H'). \tag{1}$$

Szegedy showed that, with probability 1, $\lim_{q\to\infty} \log_q(|\mathsf{hom}(H, R_q)|) = \hat{h}_R(H)$. This fact trivially implies that $(R_q)_q^\infty$ is log-convergent.

2 Our result

Let q be a prime power, and let G_q be the incidence graph of $PG(2, q)$ over the field $\mathbb{F}_q$. In other words, G_q has vertex set $P \cup L$, where P and L are the sets of points and lines of $PG(2, q)$ respectively, and an edge is drawn between a point p and a line ℓ if $p \in \ell$.

Szegedy asked whether the sequence $(G_q)_q^\infty$ is log-convergent, with the same limit as $(R_q)_q^\infty$, when q ranges over the set of primes. We answer this question affirmatively, even when q can also be a prime power. The precise statement will be given after we define a few parameters.

For any graphs H and G, let $\mathsf{inj}(H, G)$ denote the set of injective homomorphisms from H to G. For a graph H, let

$$\hat{i}(H) = \lim_{q\to\infty} \log_q(|\mathsf{inj}(H, G_q)|), \qquad \hat{h}(H) = \lim_{q\to\infty} \log_q(|\mathsf{hom}(H, G_q)|),$$

where we take $\log_q 0 = -\infty$.

Theorem 2. *For every bipartite graph H, we have $\hat{h}(H) = \hat{h}_R(H)$. In particular, $\hat{h}(H)$ exists.*

It is not a priori clear that $\hat{i}(H)$ and $\hat{h}(H)$ exist (and indeed, proving the existence of $\hat{h}(H)$ is one of the main goals of this paper). To get around this limitation, we will use a compactness argument. If there exists a bipartite graph H such that $\log_q(|\mathsf{hom}(H, G_q)|)$ does not converge to $\hat{h}_R(H)$, then there exists a subsequence of $(G_q)_q^\infty$ in which the limit exists and has a value different from $\hat{h}_R(H)$. This subsequence, in turn, admits another subsequence in which $\log_q(|\mathsf{hom}(H, G_q)|)$ and $\log_q(|\mathsf{inj}(H, G_q)|)$ converge for all bipartite graphs H. During our proof, we will assume that the limits in the definition of $\hat{i}(H)$ and $\hat{h}(H)$ are defined on this subsequence.

The fact that the sequence $(G_q)_q^\infty$ is log-convergent might not come as a big surprise, at least not if we restrict ourselves to prime values of q. A less expected result is the fact that the limit is the same as in the random bipartite graph. The reason for this is that there are certain arrangements of points and lines that are either much more common or much more rare in G_q than in the random graph.

Indeed, if G_q was truly random-like, we would expect to have $\hat{i}(H) = \hat{i}_R(H)$, from an easy computation using linearity of expectation. But $\hat{i}_R(C_4) = 4$ and $\hat{i}(C_4) = -\infty$, since every pair of distinct lines in a projective plane intersect at exactly one point. On the other hand, the point-line arrangements corresponding to Pappus's theorem (with incidence graph $\bar{P}$) and Desargues's theorem (with incidence graph $\bar{D}$) are more common in G_q than in R_q. We have $\hat{i}_R(\bar{P}) = 9$, $\hat{i}(\bar{P}) = 10$, $\hat{i}_R(\bar{D}) = 10$ and $\hat{i}(\bar{D}) = 11$.

Obviously, these arrangements are not enough to disprove Theorem 2. The graphs $K_{1,2}$, $K_{1,9}$ and $K_{1,10}$ are collapses of C_4, $\bar{P}$ and $\bar{D}$, respectively, obtained by mapping all the vertices in one of the

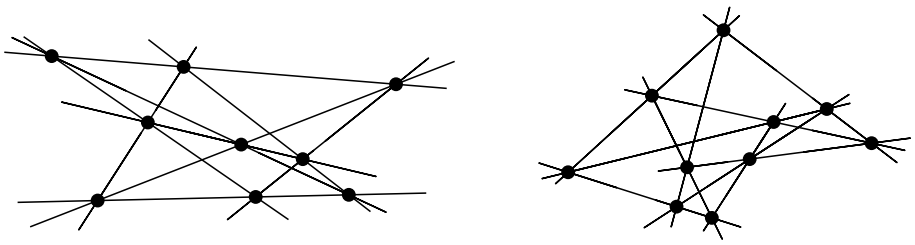


Figure 1: Point-line arrangements corresponding to Pappus's theorem (left) and Desargues's theorem (right).

bipartition classes to the center of the star. Since $\hat{i}_R(K_{1,t}) = \hat{i}(K_{1,t}) = t+2$, the value of $\hat{h}_R$ and $\hat{h}$ of our three graphs will be greater than or equal to the corresponding values of $\hat{i}_R$ and $\hat{i}$.

3 Sketch of the proof

The parameters $\hat{i}(H)$ and $\hat{h}(H)$ can be shown to satisfy the following relation:

$$\hat{h}(H) = \max_{H' \in \mathcal{C}(H)} \hat{i}(H'), \tag{2}$$

a similar relation to that between $\hat{i}_R(H)$ and $\hat{h}_R(H)$. To prove Theorem 2, we will show that $\hat{i}(H) = \hat{i}_R(H)$ holds for certain graphs H, namely those which are critical in either G_q or R_q. A graph H is said to be critical in G_q (resp. R_q) if for every $H' \in \mathcal{C}(H) \setminus \{H\}$ we have $\hat{i}(H') < \hat{i}(H)$ (resp. $\hat{i}_R(H') < \hat{i}_R(H)$).

Lemma 3. *If a bipartite graph H satisfies $\hat{i}(H) \neq \hat{i}_R(H)$, then it is neither critical in G_q nor in R_q.*

Lemma 3, together with (1) and (2) directly imply Theorem 2.

The proof of Lemma 3 works by induction on H. In a vertex-minimal counterexample, there cannot be a vertex v with degree 0 or 1, because otherwise that vertex can be removed and the terms $\hat{i}(H)$ and $\hat{i}_R(H)$ can be written in terms of those of $H - v$. In addition, if H is critical then so is $H - v$.

The minimal counterexample also cannot have a vertex v with degree 2, but the case analysis here is more complicated. We split depending on whether H is critical in G_q or in R_q, and also on the criticality of $H - v$. We have $\hat{i}(H) \leq \hat{i}(H - v)$, since in a projective plane there is exactly one line through any two points and one point at which any two lines intersect, but we cannot guarantee equality because the new point or line might be the same as one that is already in $H - v$, hence the need for a finer case analysis.

This leaves the case in which the minimum degree is at least 3. We can also assume that H is 2-connected, because otherwise we can express $\hat{i}(H)$ in terms of some smaller parts. We claim that any 2-connected graph with $\delta(H) \geq 3$ is not critical in G_q nor in R_q. The latter is easy to verify. Indeed, suppose that the two bipartition classes of H have size $a > b$. Then $\hat{i}_R(H) = 2|V(H)| - |E(H)| \leq \frac{|V(H)|}{2} \leq a < \hat{i}_R(K_{1,a})$.

If we can show that every graph with minimum degree at least 3 has $\hat{i}(H) \leq \frac{|V(H)|}{2} + 1$, then we can again bound this by $\hat{i}(K_{1,a})$. Unfortunately, while this statement is true, it does not lend itself to be proved using induction. For this reason we will apply induction on a more general statement. Let $v_2(H)$ denote the number of vertices of degree 2. We say that a subgraph S of H is a *pseudoleaf* if it is a maximal 2-connected subgraph of H and it intersects $E(H) \setminus E(S)$ in at most one vertex. Let $\ell(H)$ denote the number of pseudoleaves of H.

Lemma 4. *(a) Let H be a 2-connected graph with $\delta(H) \geq 2$, and let $v_2(H)$ be the number of vertices with degree 2. Then $\hat{i}(H) \leq \frac{|V(H)| + \max\{v_2(H), 2\}}{2}$.*

(b) Let H be a connected graph with $\delta(H) \geq 3$, and let $\ell(H)$ be the number of pseudoleaves. Then $\hat{i}(H) \leq \frac{|V(H)|+\max\{\ell(H),2\}}{2}$

A 2-connected graph with $\delta(H) \geq 3$ has $v_2(H) = 0$ and $\ell(H) = 1$, completing the proof of Lemma 3. In Lemma 4, which we prove by induction on $|E(H)|$, parts ((a)) and ((b)) give the induction step of each other. The terms $\max\{v_2(H), 2\}$ and $\max\{\ell(H), 2\}$ are necessary (see for example the constructions derived from Pappus's and Desargues's theorems). We will try to give a brief explanation for the significance of these terms.

Let H' be a 2-connected graph with $\delta(H') = 2$, and suppose that it is an induced subgraph of a strictly larger 2-connected graph H with $\delta(H) \geq 3$. How many edges of $H \setminus H'$ are incident to $V(H')$? Since H is 2-connected, there must be at least 2. But since $\delta(H) \geq 3$, every vertex of H' with degree 2 must be incident to at least one other edge in H, giving a minimum of $\max\{v_2(H'), 2\}$.

Let H' be a connected graph with $\delta(H') = 3$, and suppose that it is an induced subgraph of a strictly larger 2-connected graph H with $\delta(H) \geq 3$. How many edges of $H \setminus H'$ are incident to $V(H')$? Since H is 2-connected, there must be at least 2. But in addition, every pseudoleaf must be incident to an edge of $H' \setminus H$ at an interior vertex (that is, a vertex other than the one that connects to the rest of H'), otherwise the same subgraph would also be a pseudoleaf of H. This gives a minimum of $\max\{\ell(H'), 2\}$.

Now consider an edge-minimal counterexample H of Lemma 4. If H is not 2-connected, decompose the graph into its 2-connected blocks, and express $\hat{i}(H)$ in terms of the numbers corresponding to its blocks. If $\delta(H) \leq 2$, greedily remove any vertex of degree at most 2 until we get a subgraph with minimum degree at least 3. Use the fact that $\hat{i}(H) \leq \hat{i}(H - v) - d(v) + 2$ for any vertex v of degree at most 2, and count have many edges and vertices have been removed in the process.

Finally, if H is 2-connected and has maximum degree at least 3, then a result of Chartrand, Kaugars and Lick [1] implies that there exists an edge e such that $H - e$ is 2-connected. As $H - e$ contains at most 2 vertices of degree 2, and $\hat{i}(H) \leq \hat{i}(H - e)$, we can complete the proof.

References

[1] G. Chartrand, A. Kaugars, and D. R. Lick. "Critically n-connected graphs". In: *Proc. Amer. Math. Soc.* 32 (1972), pp. 63–68. DOI: 10.2307/2038307.

[2] A. Sidorenko. "A correlation inequality for bipartite graphs". In: *Graphs Combin.* 9.2 (1993), pp. 201–204. DOI: 10.1007/BF02988307.

[3] B. Szegedy. *Sparse graph limits, entropy maximization and transitive graphs*. 2015. DOI: 10.48550/ARXIV.1504.00858.

Seymour's second neighborhood conjecture in arbitrary orientations of a random graph

Fábio Botler[1*], Phablo F. S. Moura[2†], Tássio Naia[3‡]

[*]Programa de Engenharia de Sistemas e Computação, Instituto Alberto Luiz Coimbra de Pós-Graduação e Pesquisa em Engenharia, Univ. Federal do Rio de Janeiro, Brasil
[†]Dept. Ciência da Computação, Instituto de Ciências Exatas, Univ. Federal de Minas Gerais, Brasil
[‡]Dept. Ciência da Computação, Instituto de Matemática e Estatística, Univ. de São Paulo, Brasil

This research has been partially supported by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil – CAPES – Finance Code 001.

Abstract

A famous conjecture of Seymour, known as Second Neighborhood Conjecture (SNC), says that every orientation of a graph contains a vertex whose second neighborhood is as large as its first neighborhood. We confirm this conjecture for arbitrary orientations of the random graph $G(n,p)$, in a wide range of p. More precisely, we prove that the SNC holds asymptotically almost surely for every orientation of $G(n,p)$, for all $p = p(n)$ such that $\limsup_{n\to\infty} p < 1/4$.

1 Introduction

An *oriented graph* D is a digraph obtained by assigning directions to the edges of a simple graph G (i.e., D contains no loops, parallel arcs, nor directed cycles of length 2); we also call D an *orientation* of G. Given a natural number i, the *i-th neighborhood* of $u \in V(D)$, denoted by $N^i(u)$, is the set of vertices v for which a shortest directed path from u to v has precisely i arcs. A *Seymour vertex* (see [9]) is a vertex u for which $|N^2(u)| \geq |N^1(u)|$. In 1990, Seymour conjectured the following (see [4]).

Conjecture 1. Every oriented graph contains a Seymour vertex.

Conjecture 1, known as *Seymour's Second Neighborhood Conjecture* (SNC), has been intensively studied (see, e.g., [2, 5, 9]). Notably, it was confirmed for tournaments (orientations of cliques) by Fisher [6] and (via a purely combinatorial argument) by Havet and Thomassé [7]; it was also investigated by Cohn, Godbole, Harkness and Zhang [3] for the random digraph model in which each ordered pair of vertices is picked independently as an arc with probability $p < 1/2$.

We explore Conjecture 1 for orientations of the binomial random graph $G(n,p)$, defined as the random graph with vertex set $\{1,\dots,n\}$ in which every pair of vertices appears as an edge with probability p and edges are mutually independent. Note that a simple argument confirms Conjecture 1 when $G(n,p)$ is very sparse: indeed, for example, in every orientation of a triangle-free graph all vertices of minimum outdegree are Seymour vertices, and $G(n,p)$ is asymptotically almost surely (a.a.s.) triangle-free whenever $np \to 0$ as $n \to \infty$ (the latter is a classical result, see, e.g., [8, Theorem 3.4]). Our main result extends this observation to a much wider range of values of p.

[1]Email: fbotler@cos.ufrj.br. Research of F. B. supported by CNPq (423395/2018-1) by FAPERJ (211.305/2019), and by UFRJ (ALV 23.733).
[2]Email: phablo@dcc.ufmg.br. Research of P. F. S. M; supported by FAPEMIG (APQ-01040-21).
[3]Email: tnaia@member.fsf.org. Research of T. N. is supported by CNPq (201114/2014-3) and FAPESP (2019/04375-5, 2019/13364-7, 20/16570-4).

Theorem 2. *Let $p\colon \mathbb{N} \to (0,1)$. If $\limsup_{n\to\infty} p < 1/4$, then a.a.s. every orientation of $G(n,p)$ contains a Seymour vertex.*

Theorem 2 plays a crucial role in an ongoing work of the authors, investigating the SNC for typical orientations of $G(n,p)$.

This paper is organized as follows. In Section 2 we present the proof of Theorem 2, and in Section 3 we make a few further remarks. In this paper the usual asymptotic expressions ($\mathcal{O}(f)$, $\Omega(f)$, $\mathrm{o}(f)$, $\omega(f)$ and a.a.s. statements) are meant with respect to the parameter n as $n \to \infty$.

2 Proof of main theorem

Our proof has two main ingredients. The first one is a previous result of the authors about very sparse random graphs, which was submitted as part of an complete abstract to ETC[4], and will appear in an extended version of this article [1].

Lemma 3 ([1])**.** *If $p \in (0,1)$, and $n^4p^6 < \varepsilon/16$, then every orientation of $G(n,p)$ has a Seymour vertex with probability at least $1-\varepsilon$.*

The second ingredient is a small number of classical properties of $G(n,p)$, which are stated in Lemma 5 and are simple consequences of the following Chernoff-type concentration result.

Lemma 4 (Chernoff inequality)**.** *Let X be a binomial random variable $\mathcal{B}(N,p)$, i.e., the number of successes in N mutually independent experiments, each with success probability p. For all $t > 0$, we have $\mathbb{P}(|X - \mathbb{E}X| > t) < 2\exp\left(\frac{-t^2}{2(\sigma^2+t/3)}\right)$, where $\sigma^2 = Np(1-p)$ denotes the variance of X.*

Given a graph G and $X, Y \subseteq V(G)$, let $e(X,Y)$ be the number of edges of G with a vertex in each set, and $e(X)$ be the number of edges with both vertices in X.

Lemma 5. *Let $0 < \gamma < \lambda < 1$. For all $p\colon \mathbb{N} \to (0,1)$, a.a.s. $G = G(n,p)$ satisfies all of the following.*

(i) For every $X \subseteq V(G)$, we have

$$\left| e(X) - \binom{|X|}{2} p \right| < |X|\sqrt{3p(1-p)n} + 2n.$$

(ii) For every $X, Y \subseteq V(G)$, we have

$$\left| e(X,Y) - |X||Y|p \right| < \sqrt{6|X||Y|p(1-p)n} + 2n.$$

(iii) For every $X, Y \subseteq V(G)$, if $|X|, |Y| \le n^\gamma$, we have

$$\left| e(X,Y) - |X||Y|p \right| \le \sqrt{6|X||Y|p(1-p)n^\lambda} + 2n^\lambda.$$

(iv) For every $v \in V(G)$, we have

$$|\deg(v) - np| < \sqrt{6np(1-p)\ln n} + 2\ln n.$$

(v) For every distinct $u, v \in V(G)$ we have

$$\left||N(u) \cap N(v)| - (n-2)p^2\right| < \sqrt{6np^2(1-p^2)\ln n} + 2\ln n.$$

[4]VII Encontro de Teoria da Computação, see `https://csbc.sbc.org.br/2022/etc/`.

(vi) For every orientation D of G and every $a \in \mathbb{N}$, the set $B = \{v \in V(G) : \deg_D^+(v) < a\}$ satisfies

$$|B| \leq \frac{2}{p}(a-1) + 1 + \sqrt{12\frac{(1-p)n}{p}} + \frac{4n}{|B|p}.$$

As noted above, Lemma 5 is a consequence of the concentration of binomial random variables around their expectation. More precisely, Lemma 5 *(i)–(v)* all follow from Lemma 4 by taking a union bound over the corresponding "bad event" (that the respective random variable, such as $e(X)$, deviates much from its expectation). As for Lemma 5 *(vi)*, it follows from Lemma 5 *(i)*.

Proof of (vi) from (i). Clearly, $e(G[B]) \leq |B|(a-1)$. Therefore, by *(i)* (taking $X = B$), we have

$$|B|(a-1) \geq e(G[B]) \overset{(i)}{\geq} \binom{|B|}{2}p - |B|\sqrt{3p(1-p)n} - 2n,$$

which directly implies *(vi)*. □

A crucial step in the proof of Theorem 2 is finding a vertex w whose outneighborhood contains many vertices with large outdegree. Roughly speaking, after finding w, we note that $|N^1(w)| = O(np)$ and that $N^1(w) \cup N^2(w)$ cannot be too dense. However, since many outneighbors of w have large outdegree themselves, we conclude that $N^1(w) \cup N^2(w)$ must contain more than $2|N^1(w)|$ vertices, completing the proof.

Proof of Theorem 2. Let $p\colon \mathbb{N} \to (0,1)$. We shall prove that if $\limsup_{n\to\infty} p < 1/4$, then for all positive ε, there exists n_0 (which depends on ε and on p) such that the following holds: if $n \geq n_0$, then with probability at least $1-\varepsilon$ every orientation of $G = G(n,p)$ contains a Seymour vertex.

Fix γ and λ such that $1/2 < \gamma < \lambda < 2/3$. By choosing n_0 sufficiently large, we may assume that with probability at least $1-\varepsilon$ all statements in Lemma 5 hold. Below, we tacitly assume that n is large enough whenever necessary. Moreover, if $n^4p^6 < \varepsilon/16$, then the result follows by Lemma 3, and hence we may assume that $p \geq (\varepsilon/16)^{1/6}n^{-4/6}$. Fix an arbitrary orientation of G. We abuse notation slightly, denoting by G both the oriented and underlying graph.

Let $\beta = 1/4 - \limsup_{n\to\infty} p(n)$ and let $\alpha = 1/4 - p$. Since n is large, we may assume that $\alpha \geq \beta$ (this is the reason why n_0 depends on p). Let

$$S = \{v \in V(G) : \deg^+(v) < (1-\alpha)np/2\}.$$

We first show that a linear number of vertices of G must lie outside of S. We claim that $|S| \leq (1-\alpha/2)n$. Indeed, since $\alpha < 1/4 < 1-\alpha$, we may assume $|S| \geq \alpha n$. Since $\alpha np/4 \geq \alpha(\varepsilon/16)^{1/6}n^{2/6}$, Lemma 5 *(vi)* with $a = (1-\alpha)np/2$ implies that

$$|S| \leq \frac{2(a-1)}{p} + 1 + \sqrt{12\frac{(1-p)}{p}n} + \frac{4n}{|S|p} < (1-\alpha)n + n^{11/12} < \left(1 - \frac{\alpha}{2}\right)n.$$

So there are at least $\alpha n/2$ vertices in $\overline{S} = V(G) \setminus S$, as desired.

By Lemma 5 *(i)*, we have

$$e(\overline{S}) \geq \binom{|\overline{S}|}{2}p - |\overline{S}|\sqrt{3p(1-p)n} - 2n > \binom{|\overline{S}|}{2}p - 2|\overline{S}|\sqrt{pn} > \frac{|\overline{S}|^2 p}{3}$$

and hence, by averaging, there is a vertex $w \in \overline{S}$ for which

$$\deg^+_{\overline{S}}(w) \geq \frac{e(\overline{S})}{|\overline{S}|} \geq \frac{\alpha np}{6}. \tag{1}$$

We next show that w is a Seymour vertex in G. Let $X = N^1_G(w)$ and $Y = N^2_G(w)$, and suppose, for a contradiction, that $|Y| < |X|$. By Lemma 5 *(iv)* we have $\deg^+(w) \leq np + \sqrt{6np\ln n} + 2\ln n$. Since $p < 1/4$, we have that

$$\deg^+(w) = |X| \leq np + \sqrt{6np\ln n} + 2\ln n < \frac{n}{4} = \frac{n}{2}(1 - 2\alpha - 2p). \tag{2}$$

Also, since $np = \omega(n^{1/3})$, by Lemma 5 *(iv)* we have that

$$\deg^+(w) < np + \sqrt{6np\ln n} + 2\ln n < 2np. \tag{3}$$

Let us write $\vec{e}(A, B)$ to denote the number of arcs from A to B. Let $N = X \cap \overline{S}$. By Lemma 5 *(v)*, any two vertices u and v have at most $p^2n + \sqrt{6np^2\log n} + 2\ln n$ common neighbors, and thus at most $|N|(p^2n + \sqrt{6np^2\log n} + 2\ln n)$ arcs go from N to X. Since each vertex in $\overline{S}$ (and hence in N) has at least $(1-\alpha)np/2$ outneighbors, we have

$$\begin{aligned}\vec{e}(N, Y) &\geq |N|\frac{(1-\alpha)np}{2} - \vec{e}(N, X)\\ &\geq |N|\frac{(1-\alpha)np}{2} - |N|(p^2n + \sqrt{6np^2\ln n} + 2\ln n)\end{aligned} \tag{4}$$

The following claim will be useful.

Claim 6. *We have* $2\ln n + \sqrt{6np^2\ln n} + \sqrt{6|Y|np/|N|} = \mathrm{o}(np)$.

proof of Claim. We prove that each term is $\mathrm{o}(n)$ when divided by p. First, $\sqrt{6np^2\ln n}/p = \sqrt{6n\ln n} = \mathrm{o}(n)$. Now, since $p \geq (\varepsilon/16)^{1/6}n^{-2/3}$, we have $2\ln n/p = \mathcal{O}(n^{2/3}\ln n) = \mathrm{o}(n)$. Finally, by (3), we have $|Y| < 2np$, and since $|N| \geq \alpha np/3$, we have

$$\sqrt{\frac{|Y|6n}{|N|p}} \leq \sqrt{\frac{|Y|18}{\alpha p^2}} \leq \sqrt{\frac{36n}{\alpha p}} \leq \sqrt{\frac{36n}{\beta p}} = \mathrm{o}(n) \tag{5}$$

which completes the proof of the claim. □

Returning to the proof of the theorem, we have to cases: either $p > n^{\gamma-1}/2$ or $p \leq n^{\gamma-1}/2$.

Case 1. $p > n^{\gamma-1}/2$. By Lemma 5 *(ii)*, we have

$$\vec{e}(N, Y) \leq |N||Y|p + \sqrt{6|N||Y|pn} + 2n. \tag{6}$$

Thus, by Equations (4) and (6), we have

$$\frac{(1-\alpha)np}{2} - (p^2n + \sqrt{6np^2\ln n} + 2\ln n) \leq |Y|p + \sqrt{\frac{|Y|6np}{|N|}} + \frac{2n}{|N|}. \tag{7}$$

Since $p > n^{\gamma-1}/2$ and $\gamma > 1/2$, we have

$$\frac{2n}{|N|p} \leq \frac{6n}{\alpha np^2} = \frac{6}{\alpha p^2} < \frac{24}{\alpha n^{2\gamma-2}} = \mathrm{o}(n). \tag{8}$$

We conclude that w is a Seymour vertex, since (7) becomes

$$\begin{aligned}|Y| &\geq \frac{(1-\alpha-2p)n}{2} - \sqrt{6np^2\ln n} - \sqrt{\frac{|Y|6n}{|N|p}} - \frac{2n}{|N|p} - \frac{2\ln n}{p}\\ &\geq \frac{n}{2}(1-2\alpha-2p) \overset{(2)}{>} |X|,\end{aligned}$$

where in the second inequality we used Claim 6 and (8).

Case 2. $p \leq n^{\gamma-1}/2$. In this case (3) implies $|X| \leq n^{\gamma}$. Since $N \subseteq X$ and $|Y| < |X|$, Lemma 5 *(iii)* yields the following.

$$\begin{aligned} \vec{e}(N,Y) &\leq |N||Y|p + \sqrt{6|N||Y|pn^{\lambda}} + 2n^{\lambda} \\ &< |N||Y|p + \sqrt{6|N||Y|pn} + 2n^{\lambda} \end{aligned} \tag{9}$$

Now, from (4), using (9) instead of (6), we obtain a variation of (7) in which the term $2n/|N|$ is replaced by $2n^{\lambda}/|N|$:

$$\frac{(1-\alpha)np}{2} - (p^2 n + \sqrt{6np^2 \log n} + 2\ln n) \leq |Y|p + \sqrt{\frac{|Y|6np}{|N|}} + \frac{2n^{\lambda}}{|N|}. \tag{10}$$

We claim that $\frac{2n^{\lambda}}{|N|p} = \mathrm{o}(n)$. Indeed, since $p \geq cn^{-2/3}$ for $c = (\varepsilon/16)^{1/6}$, we have

$$\frac{2n^{\lambda}}{|N|p} \leq \frac{6n^{\lambda}}{\alpha np^2} = \frac{6n^{\lambda-1}}{\alpha p^2} < \frac{24n^{\lambda-1}}{\alpha c^2 n^{-4/3}} = \frac{24n^{\lambda+1/3}}{\alpha c^2} = \mathrm{o}(n),$$

where in the last step we used that $\lambda < 2/3$. The desired result is then obtained by solving (10) for $|Y|$ and using Claim 6 analogously to Case 1. □

3 Concluding remarks

In this paper we confirmed Seymour's Second Neighborhood Conjecture for a large family of graphs by proving that it holds a.a.s. for arbitrary orientations of the random graph $G(n,p)$, where $p = p(n)$ has limsup below $1/4$ (Theorem 2).

The basic idea of our proof is to show that, if there exists a vertex w with at most $n/4$ and at least $\alpha np/3$ outneighbors with outdegree at least $(1-\alpha)np/2$, then w is a Seymour vertex in $G(n,p)$. In order to bound $\deg^+(w)$ from above, we use the Lemma 5 *(iv)*. We believe that a tighter bound a or distinct choice of vertex is needed to improve our results. We also observe that, in our proof, we cannot drop the strict inequality in the condition on $\limsup p$, because if $\alpha n = \mathrm{o}(\sqrt{n})$ then $\overline{S}$ could be empty; moreover, if $\alpha \to 0$, then the error term $\sqrt{36n/\alpha p}$ may turn out to be $\Omega(n)$.

References

[1] F. Botler, P. F. S. Moura, and T. Naia. *The Second neighborhood conjecture in orientations of the random graph*. In preparation.

[2] G. Chen, J. Shen, and R. Yuster. "Second neighborhood via first neighborhood in digraphs". In: *Annals of Combinatorics* 7.1 (2003), pp. 15–20.

[3] Z. Cohn, A. Godbole, E. Harkness, and Y. Zhang. "The number of Seymour vertices in random tournaments and digraphs". In: *Graphs and Combinatorics* 32.5 (2016), pp. 1805–1816.

[4] N. Dean and B. Latka. "Squaring the tournament-an open problem". In: *Congressus Numerantium* (1995), pp. 73–80.

[5] D. Fidler and R. Yuster. "Remarks on the second neighborhood problem". In: *J. of Graph Theory* 55.3 (2007), pp. 208–220. DOI: https://doi.org/10.1002/jgt.20229.

[6] D. Fisher. "Squaring a tournament: A proof of Dean's conjecture". In: *J. of Graph Theory* 23.1 (1996), pp. 43–48. DOI: https://doi.org/10.1002/(SICI)1097-0118(199609)23:1<43::AID-JGT4>3.0.CO;2-K.

[7] F. Havet and S. Thomassé. “Median orders of tournaments: A tool for the second neighborhood problem and Sumner’s conjecture”. In: *J. of Graph Theory* 35.4 (2000), pp. 244–256.

[8] S. Janson, T. Łuczak, and A. Ruciński. *Random graphs*. Wiley-Interscience, New York, 2000, pp. xii+333.

[9] T. Seacrest. “The Arc-Weighted Version of the Second Neighborhood Conjecture”. In: *J. of Graph Theory* 78.3 (2015), pp. 219–228. DOI: https://doi.org/10.1002/jgt.21800.

Competitive equilibrium always exists for combinatorial auctions with graphical pricing schemes

Marie-Charlotte Brandenburg[1*], Christian Haase[2†], Ngoc Mai Tran[3‡]

[*]Max Planck Institute for Mathematics in the Sciences, Inselstr. 22, 04103 Leipzig, Germany
[†]Freie Universität Berlin, Arnimallee 3, 14195 Berlin, Germany
[‡]University of Texas at Austin, Speedway 2515, Austin TX 78712, USA

1 Introduction

In economics, an equilibrium is a state in which economic forces such as supply and demand are balanced. We illustrate the problem on the example of the *Cutlery Auction at dinner time*: Suppose there are three kids living together in a house, where there is exactly one fork, one knife and one spoon — we are in an urgent cutlery crisis. Every night, each of them gets to choose their own dish for dinner. Tonight, they choose Spaghetti, Steak and a Kiwi. Naturally, they fight over the cutlery with different preferences. The parent needs to devise a mechanism to mitigate the crisis so that everyone can go to sleep without crying. From an economics perspective, we can model this as an auction with three bidders and three items of distinct types. Each of the bidders (also referred to as *agents*) is willing to pay at most 1 dollar for their favorite combination and no other: for Kiwi, it is (knife, spoon), for Spaghetti, it is (fork, spoon), and for Steak, it is (fork, knife).

The parent's (auctioneer's) task is to set a price and suggest a distribution of items. For example, a possible price function sets 0 dollars for a single item, and 1 dollar for each pair. A possible distribution is the assignment (fork, knife)$\mapsto$ Steak, (spoon)$\mapsto$ Kiwi, $\emptyset \mapsto$ Spaghetti. In contrast to many concepts in economics, in which the auctioneer would like to maximize their own profit, the question of competitive equilibrium assigns the auctioneer with the task to make their choices in order to make all of three bidders *happy*. Informally, each of the kids participating in the auction is assumed to be happy with the assigned cutlery at the given price, if the subset S of cutlery maximizes

$$(\$ \text{ willing to pay for } S) - (\text{price of } S) .$$

among all possible subsets S of cutlery. Note that in the above example, this indeed holds and we have thus achieved a *competitive equilibrium* at the given price and allocation. The bids of the three kids and the parents' price function fit into the framework of *graphical valuations* and *graphical pricings*, in which bids and price functions are given by weight vectors on a fixed underlying graph G. Auctions with this valuation and pricing scheme have already been studied [3]. In the example above, the underlying graph is the complete graph K_3. We show that the existence of competitive equilibrium in this setting is guaranteed to exist, independently of the preferences of the three kids:

Theorem 1. *If the underlying value graph of all agents is the complete graph, then for any set of graphical valuations there exists a graphical price at which a competitive equilibrium exists.*

More precise statements can be found in Section 4. In the auctioneer's disposal is the ability to *design* the auction. We follow the setup of algorithmic auction design [3, 5], namely, that the auctioneer can restrict the class of valuations that the agents can submit as well as the class of price functions that can

[1]Email: marie.brandenburg@mis.mpg.de
[2]Email: haase@math.fu-berlin.de
[3]Email: ntran@math.utexas.edu

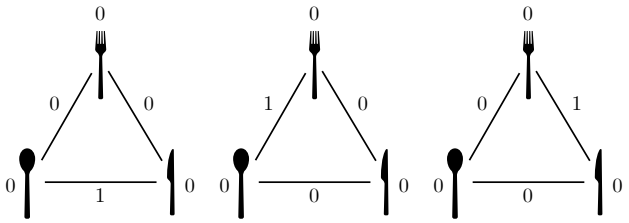


Figure 1: The graphical representation of the kids' bids in the Cutlery Auction.

be announced. The simplest pricing scheme is linear pricing, where each type of item is assigned a price p_i, and the price of a bundle S is $\sum_{i\in S} p_i$. In many combinatorial auctions, however, a competitive equilibrium with linear pricing scheme may not exist. The "Unimodularity Theorem" [1, 4, 7] gives a guarantee of a competitive equilibrium for linear prices, however it imposes strong conditions on the class of valuations allowed. This excludes many intuitive valuation profiles, such as the Cutlery Auction, which is an auction where an equilibrium under linear pricing does not exist [2]. We thus consider auctions with graphical (quadratic) pricing and graphical valuations.

2 The economic setup and its mathematical model

Consider an auction with m types of indivisible goods on sale to n agents, where $a_i^* \in \mathbb{Z}_{\geq 0}$ is the number of goods of type $i \in [m]$. We assume that each agent wants to buy at most one item of each type, even though the auctioneer might sell more than one item of this type to the group of agents. Following [3], we model the relations between the goods via a graph G on m vertices. Each vertex represents a type of goods, while edges between vertices model the existence of a relationship between types. That is, if the graph contains an edge ij, then the auctioneer allows bidders e.g. to ask for a discount when buying both an item of type i and j. We explain this in more detail below.

Given a subset of items $S \subseteq [m]$, we construct a vector $a_S \in \{0,1\}^{[m]\sqcup E(G)}$, where the first m coordinates are indexed by the vertices of G and the following coordinates are indexed by the edges of G. Explicitly, we define a_S by

$$a_{S,i} = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases} \text{ for } i \in [m], \qquad a_{S,ij} = \begin{cases} 1 & \text{if } i,j \in S \\ 0 & \text{otherwise} \end{cases} \text{ for } ij \in E(G).$$

The *polytope of the graph* G is

$$P(G) = \operatorname{conv}\left(\left\{a_S \in \{0,1\}^{[m]\sqcup E(G)} \mid S \subseteq [m]\right\}\right).$$

Example 2. In the Cutlery Auction above, we have $m = 3, a_i^* = 1$ for all $i \in [3]$, and we consider the complete graph $G = K_3$ on three nodes $V(K_3) = \{A, B, C\}$, where $A =$ Fork, $B =$ Knife and $C =$ Spoon. Using the indexing $(a_A, a_B, a_C; a_{AB}, a_{AC}, a_{BC})$, the polytope $P(K_3)$ is the convex hull

$$\begin{aligned} P(G) = \operatorname{conv}\{&(0,0,0;0,0,0),(1,0,0;0,0,0),(0,1,0;0,0,0),(0,0,1;0,0,0),\\ &(1,1,0;1,0,0),(1,0,1;0,1,0),(0,1,1;0,0,1),(1,1,1;1,1,1)\} \end{aligned}$$

where each vertex of $P(G)$ corresponds to an induced subgraph of K_3 (including the empty graph and K_3 itself).

A *graphical valuation* is an assignment $\mathrm{val}_w = \mathrm{val} : \{a_S \mid S \subseteq [m]\} \to \mathbb{R}$, $\mathrm{val}(a) = \langle w, a \rangle$, with which the agents express their bids. The value of $\mathrm{val}(a_S)$ can be viewed as the maximum that the agent is willing to pay for the set S. Such a bid can be interpreted as follows. If $w_{ij} > 0$ for some edge $ij \in E(G)$, then the agent views items of types i and j as *complementary*, that is, they are rather

interested in buying both items together than only a single one of them. The higher the value of w_{ij}, the higher is the agent's preference of buying both i and j at the same time. Similarly, if $w_{ij} < 0$, then items of types i and j are viewed as *substitutable*, that is, the agent asks for a discount when they have items i and j together. Similarly, a *graphical pricing function* $\langle p, \cdot \rangle,\ p \in \mathbb{R}^{[m] \sqcup E(G)}$ specifies the price to be paid for each bundle. They allow for an analogous interpretation to the one of valuations above: when the pair of items i and j is bought together, they require a premium of p_{ij} if $p_{ij} > 0$. If $p_{ij} < 0$, then this is a discount.

Example 3. In the Cutlery Auction, the agents' valuations are given by the weight vectors

$$w^1 = (0,0,0;1,0,0), \quad w^2 = (0,0,0;0,1,0), \quad w^b : w : [3] = (0,0,0;0,0,1),$$

i.e. the first agent has weight one for edge AB, the second agent has weight one for edge AC, the third agent has weight one for edge BC and all remaining weights are zero. The graphical price vector in this example is $p = (0,0,0;1,1,1)$.

The auctioneer enters the auction with a *bundle* $a^* \in \mathbb{Z}_{\geq 0}^{[m]}$, where a_i^* denotes the number of items of type i that the auctioneer wants to sell. In this setup, we only consider allocations that are *complete*, that is, the seller must sell all items in the bundle. Let n be the number of agents participating in the auction, and let $\pi = \pi_G$ denote the coordinate projection

$$\pi : nP(G) \cap \mathbb{Z}^{[m] \sqcup E(G)} \to \mathbb{Z}^{[m]}$$

that forgets the coordinates which correspond to edges in G.

The auction works as follows. First, the agents submit their valuations to the auctioneer, then the auctioneer announces the price p and an allocation $a^1, \ldots, a^n \in \{0,1\}^{[m] \sqcup E(G)}$, where agent b is assigned the bundle a^b and $a = \sum_{b=1}^n a^b \in \pi^{-1}(a^*)$. If all of the agents and the seller agree, then agent b gets bundle a^b and pays $\langle p, a^b \rangle$ to the seller. Note that a choice of $a \in \pi^{-1}(a^*)$ amounts to choosing values a_{ij} for each pair ij of types of goods. This specifies the number of times that an item of type i and an item of type j are sold together as a pair to an agent. Furthermore, the seller's revenue at the given price p solely depends on this choice, as $\sum_{b=1}^n \langle p, a^b \rangle = \langle p, \sum_{b=1}^n a^b \rangle = \langle p, a \rangle$.

The bare-minimum goal for the auctioneer is to compute an allocation-pricing pair such that all of the market participants can agree to (i.e. are "happy"), that is, a non-trivial economic equilibrium is reached. It is assumed that agent b is satisfied with the allocation-price pair $((a^1, \ldots, a^n), p)$ when they are allocated a bundle that maximizes their own utility at this price. At a given price p, the set of such bundles for agent b is their *demand set*

$$D(\mathrm{val}^b, p) = \operatorname*{arg\,max}_{S \subseteq [m]} \left\{ \mathrm{val}^b(a_S) - \langle a_S, p \rangle \right\},$$

where $a_S \in \{0,1\}^{[m] \sqcup E(G)}$ is the vector corresponding to the set S. Thus, agent b is satisfied with the allocation-price pair when $a^b \in D(\mathrm{val}^b, p)$. When all agents are satisfied, we have a *competitive equilibrium*.

Definition 4 (Competitive equilibrium)**.** Let $(\mathrm{val}^b : b \in [n])$ be graphical valuations with a fixed value graph G. The auction has a *competitive equilibrium* at $a \in \mathbb{Z}^{[m] \sqcup E(G)}$ and a price $p \in \mathbb{R}^{[m] \sqcup E(G)}$ if there exists an allocation $(a^1, \ldots, a^n)$ such that $a = \sum a^b$ and $a^b \in D(\mathrm{val}^b, p)$ for all $b \in [n]$.
We say that the auction has a *competitive equilibrium (CE) at a bundle* $a^* \in \mathbb{Z}_{\geq 0}^{[m]}$ *of goods* if there exists some $a \in \pi^{-1}(a^*)$ and some price $p \in \mathbb{R}^{[m] \sqcup E(G)}$ at which the auction has a competitive equilibrium.

Example 5. We continue with the example of the Cutlery Auction. We are given the bundle $a^* = (1,1,1)$. If we pick the graphical price vector $p = (0,0,0;1,1,1)$ then for each $j \in \{1,2,3\}$ we have

$$D(\mathrm{val}^b, p) \supseteq \{(0,0,0;0,0,0),\ (1,0,0;0,0,0),\ (0,1,0;0,0,0),\ (0,0,1;0,0,0)\}.$$

and additionally $D(\mathrm{val}^b, p)$ contains the indicator vector corresponding to the respective edge that agent b prefers to buy. Thus, we can decompose $a = (1,1,1;0,0,0) \in \pi^{-1}(a^*)$ by assigning one item to each agent, e.g.

$$a^1 = (1,1,0;1,0,0), \quad a^2 = (0,1,0;0,0,0), \quad a^3 = (0,0,0;0,0,0),$$

in order to achieve a competitive equilibrium at $a = a^1 + a^2 + a^3 = (1,1,1;1,0,0)$. This implies that the auction has a competitive equilibrium at the bundle $a^* = (1,1,1)$.

3 The polytope $P(G)$ and mixed regular subdivisions

We now establish the connection between the existence of a competitive equilibrium and properties of the polytope $P(G)$. Let val be a (not necessarily graphical) valuation on a polytope P, i.e. a function $\mathrm{val}: P \cap \mathbb{Z}^d \to \mathbb{R}$. For a face F of P we denote $\mathrm{lift}(F) = \left\{ \binom{a}{\mathrm{val}(a)} \mid a \in F \cap \mathbb{Z}^d \right\} \subseteq \mathbb{R}^{d+1}$. If we take the convex hull of these points in $\mathbb{R}^{d+1}$, the collection of the facets visible when we look down at the convex hull from above is called the *upper convex hull*. Its projection onto the first d coordinates is called the *regular subdivision of P induced by* val. We call $a \in \mathbb{Z}^d$ a *lifted point* if $\binom{a}{\mathrm{val}(a)}$ lies in the upper convex hull of $\mathrm{conv}(\mathrm{lift}(P))$. Let $p \in \mathbb{R}^d$ be a fixed price vector. Then $\binom{-p}{1}$ is the normal vector of a face F of the upper convex hull of $\mathrm{lift}(P(G))$, and

$$D(\mathrm{val}, p) = \operatorname*{arg\,max}_{a \in P \cap \mathbb{Z}^d} \{\mathrm{val}(a) - \langle a, p \rangle\} = \operatorname*{arg\,max}_{a \in P \cap \mathbb{Z}^d} \left\{ \left\langle \binom{a}{\mathrm{val}(a)}, \binom{-p}{1} \right\rangle \right\}$$

consists precisely of the lifted points in the face F. Let $\mathrm{val}^1, \ldots, \mathrm{val}^n$ be graphical valuations, and consider the *aggregate valuation function* $\mathrm{Val}: nP(G) \cap \mathbb{Z}^d \to \mathbb{R}$ given by

$$\mathrm{Val}(a) = \max \left\{ \sum_{b=1}^{n} \mathrm{val}^b(a) \mid \sum_{b=1}^{n} a^b = a, a^b \in P(G) \cap \mathbb{Z}^d \right\}.$$

This defines a regular subdivision on the dilated polytope $nP(G)$. By above, the faces of this subdivision are in bijection with demand sets $D(\mathrm{Val}, p)$. The regular subdivision of $nP(G)$ that is defined by Val is a *mixed regular subdivision* induced by $\mathrm{val}^1, \ldots, \mathrm{val}^n$. A face F of this mixed regular subdivision is of the form $F = \sum_{b=1}^{n} F^b$, where F^b is a face of $P(G)$. In other words, for the demand sets of the aggregate valuation holds $D(\mathrm{Val}, p) = \sum_{b=1}^{n} D(\mathrm{val}^b, p)$.

Proposition 6. *Consider an auction with n agents, anonymous graphical pricing and graphical valuations with a fixed underlying value graph G on m vertices, and $d = m + |E(G|$. Let $a \in nP(G) \cap \mathbb{Z}^d$. Then the following are equivalent:*

(i) For each set of valuations $\{\mathrm{val}^b \mid b \in [n]\}$ there exists an allocation $(a^1, \ldots, a^n)$ and a price $p \in \mathbb{R}^d$ such that $a = \sum_{b=1}^{n} a^b$ and at which a competitive equilibrium exists.

(ii) For each set of valuations $\{\mathrm{val}^b \mid b \in [n]\}$ there exists a price $p \in \mathbb{R}^d$ such that $a \in \sum_{b=1}^{n} D(\mathrm{val}^b, p)$.

(iii) For any faces $F^1, \ldots, F^n$ of $P(G)$ such that $a \in \sum_{b=1}^{n} F^b$ holds: $a \in \sum_{b=1}^{n} \mathrm{vert}(F^b)$.

Informally, if a competitive equilibrium exists for any set of valuations, then this means that a CE is guaranteed to exist: Regardless of the agents' preferences, the auctioneer can always guarantee to make everyone happy with the outcome. We note that the condition in Proposition 6 may fail, as illustrated in the following example.

Example 7. Consider an auction with 4 types of items A, B, C, D, with two items of each type, and let $G = K_4$ be the underlying value graph. The point

$$(a_A, a_B, a_C, a_D, a_{AB}, a_{AC}, a_{AD}, a_{BC}, a_{BD}, a_{CD}) = (2,2,2,2;1,1,1,1,1,1)$$

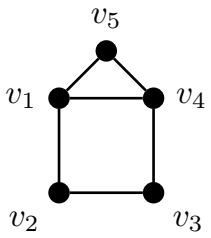


Figure 2: A value graph at which CE may fail at $a^* = (1, 1, 1, 1, 1)$.

is the sum of the midpoints of four edges but one can check that it cannot be written as the sum of any 4 lattice points of $P(K_4)$. This means that for *any* auction with graphical valuations, it is impossible to partition the set $\{A, A, B, B, C, C, D, D\}$ such that each of the $\binom{4}{2} = 6$ pairs of item types appear exactly once in some agent's allocation, even though it is possible to carry out a partition with twice as many items (that is, 4 items of each of the 4 types), where each of the six pairs of item types appears exactly twice. However, Theorem 10 below guarantees that for each set of graphical valuations, there exists a different way to partition $(2, 2, 2, 2)$ such that competitive equilibrium is achieved at a' with graphical pricing. More generally, $P(K_m)$ does not have the so-called integer decomposition property [6, p. 337] for $m \geq 4$, so this example can be extended to a series of examples for any $m \geq 4$ and $n \geq 4$.

4 Competitive equilibrium always exists for the complete graph

4.1 Everyone bids on everything

Recall that m is the number of distinct items, n is the number of bidders and π the projection of a vector with components indexed by $[m] \sqcup E(G)$ onto the first $[m]$ components. In this section, we consider the complete graph $G = K_m$, and vectors are thus of length $d = m + \binom{m}{2}$.

We begin with auctions in which the seller's bundle contains either 0 or r items of each type for a fixed $r \in \mathbb{N}$. Recall that we assume that each agent is only interested in buying at most one item per type. An important special case is the combinatorial auction where $r = 1$. We show that a competitive equilibrium in this scenario can always be achieved:

Theorem 8. *Let $a^* \in \{0, r\}^m$ and $n \geq r$. If the underlying value graph of all agents is the complete graph, then $\pi^{-1}(a^*) \neq \emptyset$ and for any set of valuations and every $a \in \pi^{-1}(a^*)$ there exists a price $p \in \mathbb{R}^d$ at which a competitive equilibrium exists.*

The proof of this theorem gives an explicit construction of how to split the bundle in question. If $r = 1$, then $a \in \pi^{-1}(a^*)$ is the characteristic vector of a disjoint union of cliques and the procedure in the proof assigns cliques to agents. A choice of $a \in \pi^{-1}(a^*)$ corresponds to a choice of connected components. This construction gives a lot of freedom to the auctioneer: The auctioneer can decide which items are being sold together and is still guaranteed to achieve a competitive equilibrium. The next example shows that even for $r = 1$, the existence of a competitive equilibrium can fail when we do not consider the complete graph as value graph.

Example 9. Let G be the graph consisting of the cycle v_1, v_2, v_3, v_4 together with an additional vertex v_5 and edges v_1v_5, v_4v_5, as shown in Figure 2. Consider the following 4 edges of the polytope $P(G)$

$$F^1 = \operatorname{conv}\Big(\big(0,1,0,0,0;0,0,0,0,0,0\big), \big(1,0,1,0,0;0,0,0,0,0,0\big)\Big)$$
$$F^2 = \operatorname{conv}\Big(\big(0,0,1,0,0;0,0,0,0,0,0\big), \big(0,1,0,1,0;0,0,0,0,0,0\big)\Big)$$
$$F^3 = \operatorname{conv}\Big(\big(0,0,0,0,1;0,0,0,0,0,0\big), \big(1,0,0,0,0;0,0,0,0,0,0\big)\Big)$$
$$F^4 = \operatorname{conv}\Big(\big(0,0,0,0,1;0,0,0,0,0,0\big), \big(0,0,0,1,0;0,0,0,0,0,0\big)\Big)$$

For $a^* = (1,1,1,1,1)$, we have $\pi^{-1}(a^*) \cap \sum_{b=1}^4 F^b = \{(1,1,1,1,1;0,0,0,0,0,0)\}$ but $\pi^{-1}(a^*) \cap \sum_{b=1}^4 \operatorname{vert}(F^b) = \emptyset$. Hence, the assumption $G = K_m$ in Theorem 8 is truly necessary.

Next, we loosen the assumption on a^* and allow arbitrary $a^* \in \mathbb{Z}^n \cap [0,n]^m$. We show that again a CE at a^* always exists, however we only construct one explicit $a \in \pi^{-1}(a^*)$ at which a competitive equilibrium is guaranteed to exist.

Theorem 10. *Let $a^* \in \{0,1,\dots,n\}^m$. If the underlying graph of all valuations is the complete graph, then there exists an $a \in \pi^{-1}(a^*)$ such that for any set of valuations there exists a price $p \in \mathbb{R}^d$ at which a competitive equilibrium exists.*

The procedure in which the bundle can be split up to achieve a competitive equilibrium is as follows: If $a^* \in \{0,1\}^m$, then auctioneer sells the entire bundle to one agent, i.e. there is one agent who gets an item of each type $i \in [m]$ such that $a_i^* = 1$. If $a^* \in \{0,1,2\}^m$, then the auctioneer sells the items in two bundles: There is one agent who will be offered one item of each type where $a_i^* > 0$. A second agent will be made an offer for all remaining items, i.e. all items such that $a_i^* = 2$. And so on.

4.2 Everything is bid on by someone

If $a^* \in \{0,1\}^m$, we can relax the condition on the valuations by allowing weight vectors $w \in \mathbb{R} \cup \{-\infty\}$, so that a bidder can express that they are not willing to take an item of a certain type under any circumstances. Under mild technical assumptions (the set of valuations should be *covering*), we show that a competitive equilibrium is guaranteed even with this more general class of valuations. Intuitively, a set of valuations is *covering* if for any item that the auctioneer wants to sell there is also an agent that is willing to take such an item under some circumstances (i.e. there is some offer that the auctioneer can make to this agent such that they will accept to take this item).

Theorem 11. *If $\mathcal{V}$ is the collection of sets of covering graphical valuations, then a competitive equilibrium with anonymous graphical pricing is guaranteed to exist at* any *bundle $a^* \in \{0,1\}^m$ for* any *auction with valuations in $\mathcal{V}$.*

References

[1] E. Baldwin and P. Klemperer. "Understanding Preferences: "Demand Types", and the Existence of Equilibrium With Indivisibilities". In: *Econometrica* 87.3 (2019), pp. 867–932. DOI: 10.3982/ecta13693.

[2] O. Candogan, A. Ozdaglar, and P. Parrilo. "Iterative Auction Design for Tree Valuations". In: *Operations Research* 63.4 (2015), pp. 751–771. DOI: 10.1287/opre.2015.1388.

[3] O. Candogan, A. Ozdaglar, and P. Parrilo. "Pricing Equilibria and Graphical Valuations". In: *ACM Transactions on Economics and Computation* 6.1 (2018). DOI: 10.1145/3175495.

[4] V. I. Danilov and G. A. Koshevoy. "Discrete convexity and unimodularity. I". In: *Advances in Mathematics* 189.2 (2004), pp. 301–324. DOI: 10.1016/j.aim.2003.11.010.

[5] T. Roughgarden and I. Talgam-Cohen. “Why Prices Need Algorithms”. In: *Proceedings of the Sixteenth ACM Conference on Economics and Computation*. Portland, Oregon, USA: Association for Computing Machinery, 2015, pp. 19–36. DOI: 10.1145/2764468.2764515.

[6] A. Schrijver. *Theory of Linear Integer Programming*. Chichester; Weinheim: John Wiley & Sons, 1998. 484 pp.

[7] N. M. Tran and J. Yu. “Product-Mix Auctions and Tropical Geometry”. In: *Mathematics of Operations Research* 44.4 (2019), pp. 1396–1411. DOI: 10.1287/moor.2018.0975.

Conway's fried potato problem: a (quadratic) algorithm leading to an optimal division for convex polygons

Antonio Cañete[1], Isabel Fernández[2], Alberto Márquez[3]

Dpto. Matemática Aplicada I and IMUS, Universidad de Sevilla

Abstract

Conway's fried potato problem is an optimization question with the following original statement [7]: given a convex body C in $\mathbb{R}^d$, which is the division of C into n subsets, by means of $n-1$ successive hyperplane cuts (that is, each cut divides only one previous subset), minimizing the largest inradius of the subsets? This question was treated in a work by A. Bezdek and K. Bezdek [1], determining the optimal value and describing an optimal division for this problem in an implicit way. In this work, we focus on the family of convex polygons, and provide an algorithm leading to the corresponding optimal value (which is of quadratic order with respect to the number of sides of the polygon).

1 Introduction

Optimization problems in $\mathbb{R}^d$ involving the classical geometric magnitudes constitute an interesting research field with a wide variety of questions to be treated. In general, these questions search for the sets minimizing one given geometric magnitude under certain restrictions regarding another magnitude. Some remarkable and well-known examples are given by *the isoperimetric problem* [12, 11], *the isodiametric inequality* [2, 13], *Jung's theorem* [9] or *Blaschke's theorem* [8]. This kind of problems can be also posed for divisions of a given set, looking for the divisions minimizing a geometric functional. For instance, one of the authors has studied the least-perimeter partitions of a planar disk into regions of prescribed area [4], as well as the divisions of a planar convex body minimizing the maximum relative diameter ([5, 3], see also [10]). A similar question appearing in literature is the so-called *Conway's fried potato problem*, where the aim is finding a division of a given convex body C into n subsets (by means of $n-1$ successive hyperplane cuts) minimizing the largest inradius of the subsets [7, Problem C1]. In other words, for each division of C into n subsets, we consider the largest value for the inradius of the subsets and search for the division minimizing that value. This problem was treated by A. Bezdek and K. Bezdek in 1995, obtaining the corresponding optimal value and a particular optimal division ([1], see also Theorem 2 below). In that work, the determination of the optimal value (and, consequently, of that optimal division) is done in a theoretical way, in the sense that the optimal value is computed implicitly and depends on a certain *rounded body* associated to C which is, in general, hard to be calculated.

In this work, we focus on the family of convex polygons, and we describe an algorithm for obtaining the optimal value associated to Conway's fried potato problem, taking into account the results in [1]. This constructive procedure, detailed in Section 3, has a geometrical core and it is based on the notion of *medial axis* of a polygon. We anticipate that it is an algorithm of quadratic order with respect to the number of sides of the polygon.

[1]Email: antonioc@us.es. Research of A. C. supported by MICINN project PID2020-118180GB-I00.
[2]Email: isafer@us.es. Research of I. F. supported by MICINN project PID2020-118137GB-I00.
[3]Email: almar@us.es. Research of A. M. supported by MICINN project PID2019-103900GB-I00.

2 Preliminaries

Throughout this work, a convex body in $\mathbb{R}^d$ will be a compact convex set with non-empty interior. Given a convex body C in $\mathbb{R}^d$, the *inradius* $I(C)$ of C is the radius of the largest ball contained in C, and the *width* $w(C)$ of C is the minimal distance between two parallel supporting hyperplanes of C.

Definition 1. Let C be a convex body in $\mathbb{R}^d$, and let $0 < \rho \leq I(C)$. The ρ-rounded body C^ρ associated to C is the union of all the balls of radius ρ contained in C.

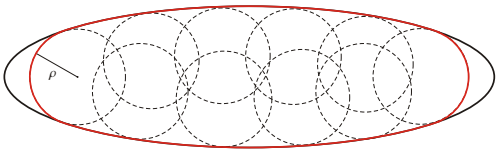


Figure 1: For a given $\rho > 0$, the ρ-rounded body (in red) of this ellipse is given by the union of all the balls of radius ρ contained in the ellipse

As stated in the Introduction, the original statement of Conway's fried potato problem focuses on divisions by *successive hyperplane cuts*. These divisions are made by using $n-1$ pieces of hyperplanes, in such a way that each piece of hyperplane divides only one of the subsets into two new subsets, see Figure 2. Note that these divisions guarantee that the corresponding subsets are convex.

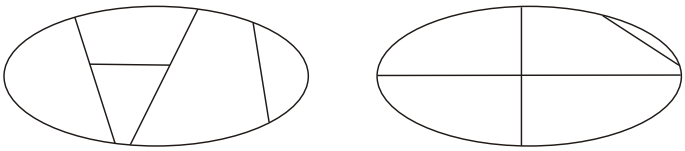


Figure 2: Two different divisions of the ellipse into five subsets, by using four successive hyperplane cuts

Theorem 2. *([1, Th. 1]) Let C be a convex body in $\mathbb{R}^d$, and $n \in \mathbb{N}$. Let P be a division of C into n subsets $C_1, \ldots, C_n$ given by $n-1$ successive hyperplanes cuts. Then,*

$$\max\{I(C_1), \ldots, I(C_n)\} \geq \rho, \tag{1}$$

where $\rho > 0$ is the unique number satisfying that

$$w(C^\rho) = 2n\rho. \tag{2}$$

Moreover, the equality in (1) *holds for the division of C given by $n-1$ parallel hyperplanes, equally spaced between the two hyperplanes bounding a slab which provides $w(C^\rho)$.*

Observe that Theorem 2 determines the optimal value for Conway's fried potato problem for each convex body C, as well as an optimal division of C (since the optimal value is attained). Unfortunately, that optimal value is given implicitly in (2), in terms of a certain rounded rounded body associated to C, which makes difficult its precise computation, in general. In Section 3 we describe an explicit algorithm for computing that optimal value for an arbitrary convex polygon. We previously define two important notions on this family, which will be used later.

Let C be a convex polygon. The *medial axis* $M(C)$ of C is defined as the set of points of C which have more than one closest side of C, see Figure 3. Equivalently, $M(C)$ is the boundary of the Voronoi diagram associated to ∂C, and so it will be composed by line segments (it is, in fact, a tree-like graph).

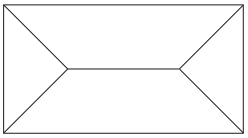


Figure 3: The medial axis of a rectangle

We point out that each segment of $M(C)$ will be associated to a pair of sides of C, and from the computational point of view, it is known that $M(C)$ can be computed in linear time with respect to the number of sides of C [6, Co. 4.5]. Moreover, we have that the ρ-rounded body C^ρ associated to C, for any $0 < \rho \leq I(C)$, will be bounded by some circular arcs of radius ρ centered in points in $M(C)$, and by some segments (each of them contained in a side of C), by construction.

Fix now a side L of a convex polygon C. The relative width of C with respect to L, denoted by $w_L(C)$, is the width of C when considering slabs parallel to the direction determined by L. In this setting, let L' be the supporting line of C, parallel to L, bounding the slab which provides $w_L(C)$. Any vertex of C contained in L' will be called an *antipodal vertex* to L. Note that we can have, at most, two antipodal vertices to L, and each of them will have two incident sides of C. For each pair of these sides (we can have, at most, three pairs), the corresponding segment of the medial axis $M(C)$ will be called an *antipodal segment* to L, see Figure 4.

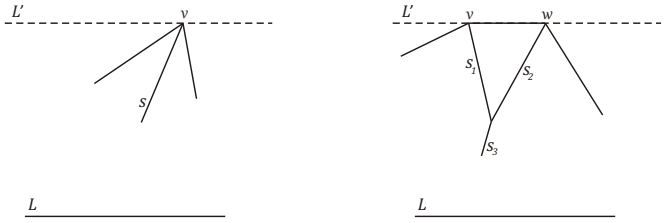


Figure 4: v is an antipodal vertex and s is an antipodal segment to L in the left-hand side figure, and v, w are antipodal vertices and s_1, s_2, s_3 are antipodal segments to L in the right-hand side figure

3 The algorithm

In this section, we will provide an algorithm for obtaining the optimal value of Conway's fried potato problem for a given convex polygon. We need some preliminary results. It is well known that the width of a convex polygon is given by a slab delimited by two supporting lines, with at least one of them containing a side of the polygon. Lemma 3 below shows that this property extends to any rounded body associated to a convex polygon. We note that d will stand for the Euclidean planar distance.

Lemma 3. *Let C be a convex polygon, and let $0 < \rho \leq I(C)$. Then, there exists a slab providing $w(C^\rho)$, such that one side of C is contained in the boundary of the slab.*

Proof. It is clear that $\rho \leq I(C^\rho) \leq w(C^\rho)/2$, and that any tangent line r to C^ρ will leave C^ρ entirely contained in one of the half-planes defined by r. Furthermore, recall that C^ρ will be composed by some circular arcs (of radius ρ) and some segments (which are pieces of sides of C). Let S be a slab determining $w(C^\rho)$, delimited by two parallel supporting lines h_1, h_2. Let x_i be a point in $h_i \cap C^\rho$, and assume that x_i lies in an arc σ_i of C^ρ, for $i = 1, 2$ (otherwise, the statement trivially follows).

If $d(x_1, x_2) = w(C^\rho)$, let B' be the ball of radius $w(C^\rho)/2$ passing through x_1, x_2. Since $\rho \leq w(C^\rho)/2$, we have two possibilities. If $\rho = w(C^\rho)/2$, then σ_1, σ_2 lie in $\partial B'$ and we can rotate the lines h_1, h_2 tangentially along $\partial B'$ (preserving their parallel character), obtaining new slabs which also provide

$w(C^\rho)$, until one of the rotating lines meets a segment of C^ρ, and so we are done. On the other hand, if $\rho < w(C^\rho)/2$, we can proceed analogously: rotate h_i along σ_i (note that $\sigma_i - \{x_i\}$ lies now in the interior of B') to obtain slabs containing C^ρ whose width is smaller than $w(C^\rho)$, yielding a contradiction, see Figure 5. Finally, if $d(x_1, x_2) > w(C^\rho)$, we can proceed similarly: by rotating h_i along σ_i as shown in

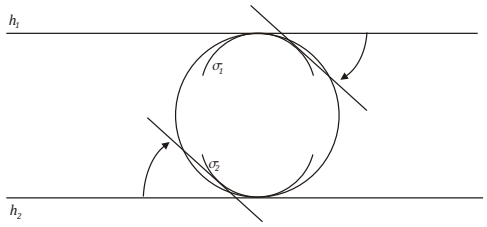


Figure 5: If $\rho < w(C^\rho)/2$, a rotation argument yields a slab containing C^ρ, with width strictly smaller than $w(C^\rho)$

Figure 6, it can be checked that the resulting slabs (which also contain C^ρ) have width smaller than $w(C^\rho)$, which also gives a contradiction. □

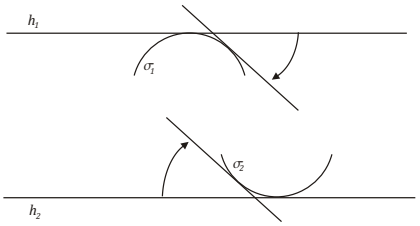


Figure 6: If $d(x_1, x_2) = w(C^\rho)$, an analogous rotation argument leads to a slab containing C^ρ, whose width is strictly smaller than $w(C^\rho)$

The following result will allow us to find the candidates for the desired optimal value, focusing on each side of the polygon.

Lemma 4. *Let C be a convex polygon, and $n \in \mathbb{N}$, $n \geq 2$. For each side L of C, there exists a unique value $\rho > 0$ such that $w_L(C^\rho) = 2n\rho$. Moreover, such a value can be computed in linear time with respect to the number of sides of C.*

Proof. The function $\rho \mapsto 2n\rho$ is strictly increasing, while the function $\rho \mapsto w_L(C^\rho)$ is continuous and decreasing (by the set inclusion property for rounded bodies), for $0 \leq \rho \leq I(C)$ (with the convention that $C^0 = C$). Moreover, for $\rho_1 = 0$, it follows that $w_L(C^{\rho_1}) = w_L(C) > 0$, and for $\rho_2 = I(C)$, we have that $w_L(C^{\rho_2}) = 2\rho_2 < 2n\rho_2$. This implies that the graphs of both functions necessarily intersect only once, thus providing the desired value.

Let us now see that the function $\rho \mapsto w_L(C^\rho)$ is piecewise affine. Assume firstly that L has only one antipodal vertex O', belonging to the sides L_1 and L_2, and call s to the antipodal segment to L (which is a segment of the medial axis $M(C)$ of C). In order to compute $w_L(C^\rho)$ for small $\rho > 0$, we only have to focus on L and on a circular arc σ of radius ρ centered at a certain point of s (in fact, $w_L(C^\rho) = d(L, L_\rho)$, being L_ρ the tangent line to σ parallel to L, see Figure 7). Observe that each point A in s will determine a unique rounded body C^ρ associated to C, with $\rho = d(A, L_1)$. Call A' to the point in L_1 such that $d(A, L_1) = d(A, A')$, and call A'' to the orthogonal projection of A on L, as Figure 7 shows. Then,

$$w_L(C^\rho) = d(A'', L_\rho) = d(A, A'') + \rho = d(A, A'') + d(A, A'). \tag{3}$$

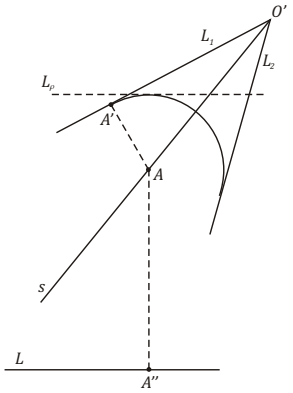


Figure 7: Computing $w_L(C^\rho)$

Let O be the intersection of the extensions of s and L, and fix a point B in s, $B \neq A$. By considering the previous geometric construction for B, and for the associated points B' and B'', it follows that

$$\frac{d(O',A)}{d(O',B)} = \frac{d(O',A')}{d(O',B')} = \frac{d(A,A')}{d(B,B')}, \qquad \frac{d(O,A)}{d(O,B)} = \frac{d(O,A'')}{d(O,B'')} = \frac{d(A,A'')}{d(B,B'')},$$

since the corresponding pairs of triangles are equivalent. Therefore, $d(A,A'') = \lambda\,(d(O,O') - d(O',A))$ and $d(A,A') = \mu\, d(O',A)$, where $\lambda = d(B,B'')/d(O,B)$ and $\mu = d(B,B')/d(O',B)$, which implies that

$$d(A,A'') + d(A,A') = (\mu - \lambda)\, d(O',A) + \lambda\, d(O,O'), \tag{4}$$

which is an affine expression with respect to $d(O',A)$. Finally, taking into account that $d(O',A) = d(A,A')/\mu$, it follows that (4) is an affine expression with respect to $d(A,A')$, and so the value $w_L(C^\rho)$, for small $\rho > 0$, can be expressed as an affine function on ρ.

The computation of $w_L(C^\rho)$ for larger ρ will be done by considering the segments of $M(C)$ adjacent to s (and possibly the subsequent ones) and proceeding analogously, until ρ reaches $I(C)$. This will lead to a piecewise affine expression of $w_L(C^\rho)$ (since each considered segment of $M(C)$ will give an affine piece of the function), which can be computed in linear time with respect to the number of sides of the polygon (recall that $M(C)$ can be also computed with that computational cost).

On the other hand, if L has two antipodal vertices, we have that $\rho \mapsto w_L(C^\rho)$ will be constant for small $\rho > 0$ (since $w_L(C^\rho) = w_L(C)$), and piecewise affine afterwards by reasoning as in the previous case. This computation can be also done in linear time with respect to the number of sides of C.

All these considerations imply that the graphs of the functions $\rho \mapsto 2n\rho$ and $\rho \mapsto w_L(C^\rho)$ can be computed in linear time, and also the corresponding intersection point, concluding the proof. □

Our Theorem 5 follows from the previous results and establishes that the optimal value for this problem can be found in quadratic time.

Theorem 5. *Let C be a convex polygon and let $n \in \mathbb{N}$. Then, the optimal value for the corresponding Conway's fried potato problem, which is given in* (2)*, can be computed in quadratic time with respect to the number of sides of C.*

Proof. For each side L of C, consider the unique value $\rho_L > 0$ such that $w_L(C^{\rho_L}) = 2n\rho_L$, which can be determined in linear time, by using Lemma 4. When repeating the process for all the sides of C, we will obtain a family $\{\rho_L : L \text{ is a side of C}\}$ in quadratic time. This family will necessarily contain the desired optimal value, taking into account Theorem 2 and Lemma 3. We claim that the optimal value is $\rho_1 := \min\{\rho_L : L \text{ is a side of C}\}$, associated to a side L_1 of C. On the one hand, $w(C^{\rho_1}) \leq w_{L_1}(C^{\rho_1})$,

by definition. And on the other hand, by using Lemma 3 again we have that there exists a side L' of C with associated value $\rho' \in \mathcal{C}$ (by Lemma 4) such that

$$w(C^{\rho_1}) = w_{L'}(C^{\rho_1}) \geq w_{L'}(C^{\rho'}) = 2n\rho' \geq 2n\rho_1 = w_{L_1}(C^{\rho_1}),$$

where we are using that $\rho_1 \leq \rho'$, and so $C^{\rho'} \subseteq C^{\rho_1}$. Thus, $w(C^{\rho_1}) = w_{L_1}(C^{\rho_1}) = 2n\rho_1$, and so ρ_1 is the desired optimal value, by the uniqueness property from Theorem 2. As taking minimum can be done in linear time in this case, the statement holds. □

Remark 6. *We note that for a given planar convex body C, we can approximate its optimal value for this problem by considering an appropriate convex polygon C', close enough to C, and applying the described procedure to C'.*

References

[1] A. Bezdek and K. Bezdek. "A solution of Conway's fried potato problem". In: *Bull. London Math. Soc.* 27.5 (1995), pp. 492–496. DOI: 10.1112/blms/27.5.492.

[2] L. Bieberbach. "Über eine Extremaleigenschaft des Kreises." In: *Jber. Deutsch. Math. Verein.* 24 (1915), pp. 247–250.

[3] A. Cañete and B. González Merino. "On the isodiametric and isominwidth inequalities for planar bisections". In: *Rev. Mat. Iberoam.* 37.4 (2021), pp. 1247–1275. DOI: 10.4171/rmi/1225.

[4] A. Cañete and M. Ritoré. "Least-perimeter partitions of the disk into three regions of given areas". In: *Indiana Univ. Math. J.* 53.3 (2004), pp. 883–904. DOI: 10.1512/iumj.2004.53.2489.

[5] A. Cañete, U. Schnell, and S. Segura Gomis. "Subdivisions of rotationally symmetric planar convex bodies minimizing the maximum relative diameter". In: *J. Math. Anal. Appl.* 435.1 (2016), pp. 718–734. DOI: 10.1016/j.jmaa.2015.10.053.

[6] F. Chin, J. Snoeyink, and C. A. Wang. "Finding the medial axis of a simple polygon in linear time". In: *Discrete Comput. Geom.* 21.3 (1999), pp. 405–420. DOI: 10.1007/PL00009429.

[7] H. T. Croft, K. J. Falconer, and R. K. Guy. *Unsolved problems in geometry*. Problem Books in Mathematics. Unsolved Problems in Intuitive Mathematics, II. Springer-Verlag, New York, 1991, pp. xvi+198. DOI: 10.1007/978-1-4612-0963-8.

[8] H. G. Eggleston. *Convexity*. Cambridge Tracts in Mathematics and Mathematical Physics, No. 47. Cambridge University Press, New York, 1958, pp. viii+136.

[9] H. Jung. "Über die kleinste Kugel, die eine räumliche Figur einschliesst". In: *J. Reine Angew. Math.* 123 (1901), pp. 241–257. DOI: 10.1515/crll.1901.123.241.

[10] C. Miori, C. Peri, and S. Segura Gomis. "On fencing problems". In: *J. Math. Anal. Appl.* 300.2 (2004), pp. 464–476. DOI: 10.1016/j.jmaa.2004.06.050.

[11] R. Osserman. "The isoperimetric inequality". In: *Bull. Amer. Math. Soc.* 84.6 (1978), pp. 1182–1238. DOI: 10.1090/S0002-9904-1978-14553-4.

[12] E. Schmidt. "Über eine neue Methode zur Behandlung einer Klasse isoperimetrischer Aufgaben im Grossen". In: *Math. Z.* 47 (1942), pp. 489–642. DOI: 10.1007/BF01180976.

[13] P. Uryson. "Mittlere Breite und Volumen der konvexen Körper im n-dimensionalen Raume". In: *Matem. Sb. SSSR* 31 (1924), pp. 477–486.

Enumeration of chordal planar graphs and maps

Jordi Castellví[1*], Marc Noy[2*,†,‡], Clément Requilé[3*,†]

[*]Department of Mathematics of the Universitat Politècnica de Catalunya, Barcelona, Spain
[†]Institute of Mathematics (IMTech) of the Universitat Politècnica de Catalunya
[‡]Center for Mathematical Research (CRM), Barcelona, Spain

A full version of this work can be found at `https://arxiv.org/abs/2202.13340`

Abstract

We determine the number of labelled chordal planar graphs with n vertices, which is asymptotically $c_1 \cdot n^{-5/2}\gamma^n n!$ for a constant $c_1 > 0$ and $\gamma \approx 11.89235$. We also determine the number of rooted simple chordal planar maps with n edges, which is asymptotically $c_2 n^{-3/2}\delta^n$, where $\delta = 1/\sigma \approx 6.40375$, and σ is an algebraic number of degree 12. The proofs are based on combinatorial decompositions and singularity analysis. Chordal planar graphs (or maps) are a natural example of a subcritical class of graphs in which the class of 3-connected graphs is relatively rich. The 3-connected members are precisely chordal triangulations, those obtained starting from K_4 by repeatedly adding vertices adjacent to an existing triangle.

1 Introduction

A graph is chordal if every cycle of length greater than three contains at least one chord, which is an edge connecting non-consecutive vertices of the cycle. Chordal graphs have been much studied in structural graph theory and graph algorithms (see for instance [10]), but much less from the point of view of enumeration. It is known that the asymptotic number of labelled chordal graphs with n vertices is $\binom{n}{n/2}2^{n^2/4}$; an explanation for this estimate is that as n goes to infinity almost all chordal graphs with n vertices are split, that is, the vertex set can be partitioned into a clique and an independent set [1]. See also [13] for results on the exact counting of chordal labelled graphs.

On the other hand, there has been much work on counting planar graphs and related classes of graphs since the appearance of [8]. Here we focus on planar graphs that are at the same time chordal. To count them we use, as in [8], the canonical decomposition of graphs into k-connected components for $k = 1, 2, 3$. The starting point is the enumeration of 3-connected chordal planar graphs: these are precisely the chordal triangulations, which suitably rooted are in bijection with ternary trees. Then we use the decomposition of 2-connected graphs into 3-connected components. An important difference with the class of all planar graphs is that one cannot compose more than two graphs in series since otherwise a chordless cycle is created.

A more significant difference is that the class of chordal planar graphs is *subcritical* (a concept defined below), instead of being *critical* as the class of all planar graphs: this is reflected by the polynomial term $n^{-5/2}$ of the asymptotic estimates for the number of graphs in the class [4], as opposed to $n^{-7/2}$ for all planar graphs [8]. Thus we have a natural example in which the class of 3-connected graphs is relatively rich, yet the class is subcritical. In this context, being subcritical means the following. Let $C(x)$ and $B(x)$ be the generating functions, respectively, of connected and 2-connected graphs in a class that is closed under connected components and blocks, and ρ_C and ρ_B are the respective radius of convergence.

[1]Email: jordi.castellvi@upc.edu
[2]Email: marc.noy@upc.edu
[3]Email: clement.requile@upc.edu

The class is subcritical if $\rho_C C'(\rho_C) < \rho_B$, and is critical if equality holds. Subcriticality has important implications on the structure of a class of graphs. Intuitively, subcritical classes are "tree-like" in some sense exhibited for instance by the fact that the global structure is essentially determined by the block-decomposition tree, while the blocks themselves have expected constant size.

Our first result is the following.

Theorem 1. *Let g_n be the number of labelled chordal planar graphs with n vertices, c_n those which are connected, and b_n those which are 2-connected. Then as $n \to \infty$ we have*

1. $g_n \sim g \cdot n^{-5/2}\gamma^n n!, \qquad \gamma \approx 11.89235,\ g \approx 0.00027205$
2. $c_n \sim c \cdot n^{-5/2}\gamma^n n!, \qquad c \approx 0.00027194,$
3. $b_n \sim b \cdot n^{-5/2}\gamma_b^n n!, \qquad \gamma_b \approx 10.76897,\ b \approx 0.00016215.$

Our second result is about rooted maps. A rooted map is a connected planar multigraph with a fixed embedding in the plane in which an edge (the root edge) is distinguished and directed. Rooted maps where first enumerated by Tutte [12] and since then they have been the object of much study (see [11] for definitions on maps and an overview on their enumeration). We only consider simple maps (those with no loops and multiple edges) since they are the natural objects with respect to the property of being chordal.

Theorem 2. *Let M_n be the number of rooted chordal simple planar maps with n edges, and B_n those which are 2-connected. Then as $n \to \infty$ we have*

1. $B_n \sim b \cdot n^{-3/2} \cdot \sigma_b^{-n}, \qquad$ *with* $b \approx 0.071674$ *and* $\sigma_b^{-1} \approx 3.65370,$
2. $M_n \sim m \cdot n^{-3/2} \cdot \sigma^{-n}, \qquad$ *with* $m \approx 0.12596$ *and* $\sigma^{-1} \approx 6.40375.$

The proof is again based on the structure of 3-connected chordal maps and is omitted for lack of space. As opposed to the class of general maps, the class of simple chordal maps is again subcritical. This is reflected in the term $n^{-3/2}$ instead of the usual $n^{-5/2}$ for classes of planar maps. Other natural subcritical classes are outerplanar maps [7] and series-parallel maps [5], but these two classes do not contain 3-connected graphs.

2 Generating functions of chordal planar graphs

2.1 3-connected graphs

Chordal planar graphs that are 3-connected are in fact chordal triangulations (also called stacked triangulations), i.e. those obtained from K_4 by repeatedly inserting a vertex in the interior of an existing triangular face and making it adjacent to the three vertices of the boundary. Note that chordal triangulations are planar. Let then $T(z)$ be the generating function counting labelled chordal triangulations rooted at a directed edge, where z marks the number of vertices minus two. Notice that 3-connected chordal maps rooted at a triangle are in bijection with (plane) ternary trees rooted at a leaf. Thus, we consider next the generating function counting ternary trees rooted at a leaf, where z marks internal nodes. It is given by

$$S(z) = z(1 + S(z))^3. \tag{1}$$

And following the bijection, we clearly have $T(z) = zS(z)/2$. The unrooted version of $T(z)$ is needed later and can be computed by integrating $S(z)$.

Lemma 3. *Let $U(z)$ be the generating function counting unrooted labelled chordal triangulations, where z marks all the vertices. Then*

$$U(z) = \frac{z^3}{24}(S(z) - S(z)^2). \tag{2}$$

2.2 2-connected graphs

First we consider *networks*, that are 2-connected labelled chordal planar graphs rooted at a directed edge e so that the endpoints of e are not labelled.

Lemma 4. *Let $E(x,y)$ be the generating function of networks, where x marks vertices and y edges. Then*

$$E(x,y) = y\exp\left(xE(x,y)^2 + \frac{T\left(xE(x,y)^3\right)}{E(x,y)}\right). \tag{3}$$

Proof. Following the classical decomposition (see for instance [9]), networks are parallel compositions of series compositions together with their 3-connected components. A 3-connected component in a network is a chordal triangulation rooted at an edge, in which the side of every triangle save the root edge of the triangulation is replaced by a network, corresponding to the term $T(xE(x,y)^3)/E(x,y)$. Furthermore, at most two networks can be composed in series, as otherwise there would be an induced cycle of length at least four. Thus, a series composition can be encoded as a triangle rooted at an edge, in which each non-root edge is replaced by a network, corresponding to the term $xE(x,y)^2$. □

If $B(x,y)$ is the generating function of 2-connected chordal planar graphs, including the single edge, then as usual we have

$$E(x,y) = \frac{2y}{x^2}B_y(x,y), \qquad \text{and thus} \quad B(x,y) = \frac{x^2}{2}\int \frac{E(x,y)}{y}dy. \tag{4}$$

In order to compute this integral we use the so-called *dissymmetry theorem* for enumerating classes of graphs encoded by trees as presented in [2]. A class of graphs $\mathcal{A}$ is said to be *tree-decomposable* if for each graph $\gamma \in \mathcal{A}$ we can associate in a unique way a tree $\tau(\gamma)$ whose nodes are distinguishable (for instance, by using the labels). Let $\mathcal{A}_\bullet$ denote the class of graphs in $\mathcal{A}$ where a node of $\tau(\gamma)$ is distinguished. Similarly, $\mathcal{A}_{\bullet-\bullet}$ is the class of graphs in $\mathcal{A}$ where an edge of $\tau(\gamma)$ is distinguished, and $\mathcal{A}_{\bullet\to\bullet}$ those where an edge $\tau(\gamma)$ is distinguished and given a direction. The dissymmetry theorem allows us to express the class of unrooted graphs in $\mathcal{A}$ in terms of the rooted classes.

Lemma 5. *Let $\mathcal{A}$ be a tree-decomposable class. Then*

$$\mathcal{A} + \mathcal{A}_{\bullet\to\bullet} \simeq \mathcal{A}_\bullet + \mathcal{A}_{\bullet-\bullet},$$

where $\simeq$ is a bijection preserving the number of nodes.

If follows that we can write the series $A(z)$ of $\mathcal{A}$ in terms of the series of the other classes. If the encoding trees have no adjacent nodes of the same type (as in our applications) then $\mathcal{A}_{\bullet\to\bullet}$ is twice $\mathcal{A}_{\bullet-\bullet}$ and we have $A(z) = A_\bullet(z) - \mathcal{A}_{\bullet-\bullet}(z)$.

We encode a 2-connected chordal graph with a canonical tree, whose nodes are of the following types: e (edge), s (series), and t (triangulation). An example of the decomposition of a 2-connected chordal planar graph and its associated tree is illustrated by Figure 1. Notice that the edges in the tree can only be of type $e-s$ or $e-t$. We denote by $R_e(x)$, $R_s(x)$, $R_t(x)$, $R_{e-s}(x)$, $R_{e-t}(x)$ the generating functions encoding trees rooted at the corresponding specific type of node or edge. By symmetry we have $R_{e-s}(x) = R_{e\to s}(x) = R_{s\to e}(x)$, and the same goes for $R_{e-t}(x)$. Encoding those generating functions in term of networks yields

$$R_s(x) = \frac{x^3E^3}{6}, \qquad R_e(x) = \frac{x^2}{2}\left(E - xE^2 - \frac{T(xE^3)}{E}\right), \qquad R_t(x) = \frac{U(xE^3)}{E^6},$$

$$R_{e-s}(x) = \frac{x^3}{2}E^2\left(E-1\right), \qquad R_{e-t}(x) = \frac{x^2}{2}\frac{T(xE^3)(E-1)}{E}.$$

Thus, from Lemma 5 we have $B(x) = B(x,1) = R_s(x) + R_e(x) + R_t(x) - R_{e-s}(x) - R_{e-t}(x)$, and $B(x)$ can be written in terms of $E = E(x,1)$ and $S(xE^3)$ only by setting $z = xE^3$ in Equation (2) and in the relation $T(z) = zS(z)/2$, as follows.

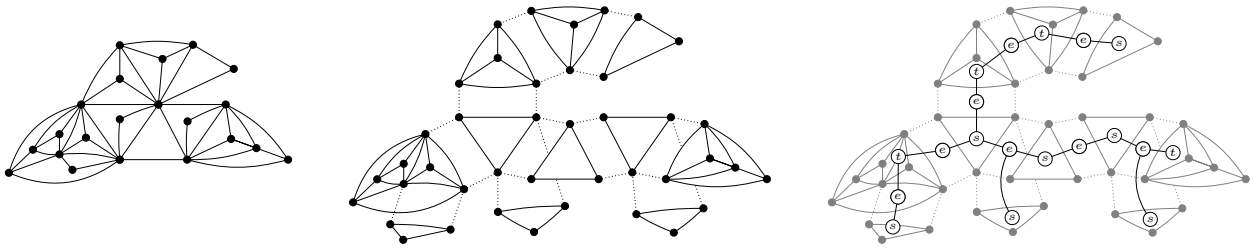


Figure 1: Left is an example of a 2-connected chordal planar graph. Middle is its decomposition into series and 3-connected components. Right is the tree associated with the decomposition.

Lemma 6. *Set $y = 1$, and let $E = E(x) = E(x, 1)$. Then*

$$B(x) = \frac{x^2}{2}\left(E - \frac{xE^3}{12}\left(S\left(xE^3\right)^2 + 5S\left(xE^3\right) + 8\right)\right). \tag{5}$$

2.3 Connected and arbitary graphs

Since chordality is preserved under the decomposition into blocks, the generating function $C^\bullet(x, y)$ of vertex rooted chordal connected graphs is given by (see [9])

$$C^\bullet(x, y) = xe^{B'(C^\bullet(x,y),y)}. \tag{6}$$

Finally, the generating function of all chordal planar graphs is $G(x, y) = e^{C(x,y)}$.

3 Proof of Theorem 1

2-connected graphs. Using $T(z) = zS(z)/2$ with $z = x(1 + F)^3$ and setting $y = 1$, $F = F(x) = E(x) - 1$ and $S = S(x(1 + F)^3)$, we transform Equations (1) and (3) into a system amenable to the so-called Drmota-Lalley-Wood's theorem [3, Theorem 2.33] (with $u = 1$), as follows:

$$\begin{aligned} F &= \exp\left(x(1+F)^2 + \frac{x(1+F)^2 S}{2}\right) - 1, \\ S &= x(1+F)^3(1+S)^3. \end{aligned} \tag{7}$$

Let $\Phi(x, S, F)$ and $\Psi(x, S, F)$ be the right-hand side of the first and second equation in (7), respectively. Those functions are entire and define a system with a strongly connected dependency graph between variables S and F. Furthermore, both have non-negative coefficients and vanish at $x = 0$, while they satisfy $\Phi(x, 0, 0) \neq 0$ and $\Psi(x, 0, 0) \neq 0$, but also $\Phi_x(x, S, F) \neq 0$ (where $\Phi_x = \partial\Phi/\partial x$) and $\Psi_x(x, S, F) \neq 0$. Finally, System (7) extended by its Jacobian admits a solution that is non-zero. It is given by the following approximations:

$$\rho_b \approx 0.092859, \quad E_0 = E(\rho_b) = 1 + F(\rho_b) \approx 1.16454, \quad S_0 = S(\rho_b E_0^3) \approx 0.41919. \tag{8}$$

Thus the hypotheses of [3, Theorem 2.33] are verified. This implies in particular that ρ_b is the unique dominant singularity of the function $E(x)$, i.e. on the boundary of the disk of convergence, and that $E(x)$ admits the following analytic continuation to a Δ-domain D at $x = \rho_b$ (see [6] for a precise definition):

$$E(x) = E_0 - E_1\sqrt{1 - \frac{x}{\rho_b}} + O\left(1 - \frac{x}{\rho_b}\right) \qquad \text{for } x \sim \rho_b \text{ and } x \in D, \tag{9}$$

where $E_1 > 0$ is computed next. Since S is itself a function of x and F, (7) can be re-written as $F = \Theta(x, F)$, where Θ is analytic at (ρ_b, F_0) with $F_0 = F(\rho_b)$. One checks that $\Theta_x(\rho_b, F_0) \neq 0$, $\Theta_F(\rho_b, F_0) = 0$ and $\Theta_{FF}(\rho_b, F_0) \neq 0$. And we can apply [6, Lemma VII.3] (see also [3, Remark 2.20]) to obtain

$$E_1 = F_1 = \sqrt{\frac{2\rho_b \Theta_x(\rho_b, F_0)}{\Theta_{FF}(\rho_b, F_0)}} \approx 0.092354. \tag{10}$$

Furthermore, [3, Theorem 2.33] implies a similar result for $S = S(x(1+F)^3)$. Note also that $\rho_b E_0^3 = 0.14665 < 4/27$, where $4/27$ is the dominant singularity of $S(z)$. This implies that the composition scheme $S(xE(x)^3)$ is subcritical.

With those results at hand, we can finally consider the generating function $B(x)$. Given the expression (5) and the fact that the scheme is subcritical, the dominant singularity of $B(x)$ is the same as that of $E(x)$ and it is furthermore unique. We show next that $B(x)$ admits a singular expansion at $z = \rho_b$ similar to $E(x)$. First we extend the system (7) so that it includes the variable y:

$$\begin{aligned} F &= y \exp\left(x(1+F)^2 + \frac{x(1+F)^2 S}{2}\right) - 1, \\ S &= x(1+F)^3(1+S)^3, \end{aligned} \tag{11}$$

where now $F = F(x, y)$. By [3, Theorem 2.33] (setting $u = y$) there exist three functions $\rho_b(y)$, $f_0(y)$ and $f_1(y)$ analytic in a neighbourhood W of 1 such that for $y \in W$ and $x \sim \rho_b(y)$ with $|\arg(x - \rho_b(y))| \neq 0$ the following singular expansion holds

$$E(x, y) = 1 + F(x, y) = 1 + f_0(y) - f_1(y)\sqrt{1 - \frac{x}{\rho_b(y)}} + O\left(1 - \frac{x}{\rho_b(y)}\right), \tag{12}$$

where $\rho_b(1) = \rho_b$, $1 + f_0(1) = E_0$ and $f_1(1) = E_1$. From there, applying [3, Theorem 2.30] to (4) and setting $y = 1$, we obtain that $B(x) = B(x, 1)$ admits an analytic continuation of the form

$$B(x) = B_0 - B_2\left(1 - \frac{x}{\rho_b}\right) + B_3\left(1 - \frac{x}{\rho_b}\right)^{3/2} + O\left(\left(1 - \frac{x}{\rho_b}\right)^2\right) \qquad \text{for } x \sim \rho_b \text{ and } x \in D.$$

The above coefficients can be computed by substituting into (5) the expansions of $E(x)$ and $S(xE(x)^3)$ when $x = \rho_b(1 - X^2)$, with $X = \sqrt{1 - x/\rho_b}$. This gives $B_0 \approx 0.0044796$, $B_2 \approx 0.0085328$ and $B_3 \approx 0.00038321$. The estimate on b_n follows from the *transfer theorem* (see [6]), with $b = 3B_3/(4\sqrt{\pi}) \approx 0.00016215$. □

Connected and arbitrary graphs. The composition scheme (6) is subcritical because $B''(\rho_b) \to \infty$. This means in particular that the singularities of $C^\bullet(x)$ come from a branch point and not from those of $B(x)$. They must hence be solution of the following system:

$$\rho = \tau e^{-B'(\tau)} \quad \text{and} \quad \tau B''(\tau) = 1, \qquad \text{with } \tau = C^\bullet(\rho) < \rho_b.$$

To find such a solution, one must first compute $E'(x)$ and $E''(x)$ and then $B''(x)$. This is a routine but lengthy computation, best solved numerically[4] together with equations (3) and (4), and which gives the following approximate solutions:

$$\tau \approx 0.092859 \quad \text{and} \quad E(\tau, 1) \approx 1.1644565. \tag{13}$$

Then by inversion the singularity is given by

$$\rho = \tau e^{-B'(\tau)} \approx 0.084088.$$

[4]We used `Maple 2021` for those computations.

A similar argument as for 2-connected graphs shows that $C^\bullet(x)$ can be extended analytically to a Δ-domain D' at $x = \rho$. As $C(x)$ is simply the integral of $C^\bullet(x)/x$, it admits the same analytic continuation in D' (see [9, Proposition 3.10.(1)]), that is:

$$C(x) = C_0 - C_2\left(1 - \frac{x}{\rho}\right) + C_3\left(1 - \frac{x}{\rho}\right)^{3/2} + O\left(\left(1 - \frac{x}{\rho}\right)^2\right) \qquad \text{for } x \sim \rho \text{ and } x \in D'.$$

Furthermore we have $C_0 \approx 0.00037470$, $C_2 = \tau$ and $C_3 \approx 0.00027194$. The claimed asymptotic estimate then follows by the transfer theorem.

The same applies to the generating function $G(x)$ counting all chordal planar graphs, and we have $G(x) = e^{C_0}(1 - C_2(1 - x/\rho) + C_3(1 - x/\rho)^{3/2} + O((1 - x/\rho)^2))$, with $G_3 = C_3 e^{C_0} \approx 0.00027205$. □

Acknowledgement. We gratefully acknowledge earlier discussions on this project with Erkan Narmanli. M.N. was supported by grants MTM2017-82166-P and PID2020-113082GB-I00, the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M). C.R. was supported by the grant Beatriu de Pinós BP2019, funded by the H2020 COFUND project No 801370 and AGAUR (the Catalan agency for management of university and research grants), and the grant PID2020-113082GB-I00 of the Spanish ministry of Science and Innovation.

References

[1] E. A. Bender, L. B. Richmond, and N. C. Wormald. "Almost all chordal graphs split". In: *J. Austral. Math. Soc. Ser. A* 38.2 (1985), pp. 214–221. DOI: 10.1017/S1446788700023077.

[2] G. Chapuy, É. Fusy, M. Kang, and B. Shoilekova. "A complete grammar for decomposing a family of graphs into 3-connected components". In: *Electron. J. Combin.* 15.1 (2008), Research Paper 148, 39.

[3] M. Drmota. *Random trees*. SpringerWienNewYork, Vienna, 2009, pp. xviii+458. DOI: 10.1007/978-3-211-75357-6.

[4] M. Drmota, É. Fusy, M. Kang, V. Kraus, and J. Rué. "Asymptotic study of subcritical graph classes". In: *SIAM J. Discrete Math.* 25.4 (2011), pp. 1615–1651. DOI: 10.1137/100790161.

[5] M. Drmota, M. Noy, and B. Stufler. "Cut vertices in random planar maps". In: *31st International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms*. Ed. by M. Drmota and C. Heuberger. Vol. 159. Dagstuhl, Germany: Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2020, Art. No. 10, 18. DOI: 10.4230/LIPIcs.AofA.2020.10.

[6] P. Flajolet and R. Sedgewick. *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009, pp. xiv+810. DOI: 10.1017/CBO9780511801655.

[7] I. Geffner and M. Noy. "Counting outerplanar maps". In: *Electron. J. Combin.* 24.2 (2017), Paper No. 2.3, 8. DOI: 10.37236/6249.

[8] O. Giménez and M. Noy. "Asymptotic enumeration and limit laws of planar graphs". In: *J. Amer. Math. Soc.* 22.2 (2009), pp. 309–329. DOI: 10.1090/S0894-0347-08-00624-3.

[9] O. Giménez, M. Noy, and J. Rué. "Graph classes with given 3-connected components: asymptotic enumeration and random graphs". In: *Random Structures & Algorithms* 42.4 (2013), pp. 438–479. DOI: 10.1002/rsa.20421.

[10] M. C. Golumbic. *Algorithmic graph theory and perfect graphs*. Computer Science and Applied Mathematics. Academic Press, New York-London-Toronto, Ont., 1980, pp. xx+284.

[11] G. Schaeffer. "Planar maps". In: *Handbook of enumerative combinatorics*. Ed. by M. Bóna. Discrete Math. Appl. (Boca Raton). CRC Press, Boca Raton, FL, 2015, pp. 335–395. DOI: doi.org/10.1201/b18255.

[12] W. T. Tutte. “A census of planar maps”. In: *Canadian J. Math.* 15 (1963), pp. 249–271. DOI: 10.4153/CJM-1963-029-x.

[13] N. C. Wormald. “Counting labelled chordal graphs”. In: *Graphs Combin.* 1.2 (1985), pp. 193–200. DOI: 10.1007/BF02582944.

Characterizing the extremal families in Erdős–Ko–Rado theorems

Gilad Chase[1*], Neta Dafni[2*], Yuval Filmus[3*], Nathan Lindzey[4†]

[*]Department of Computer Science, Technion — Israel Institute of Technology
[†]Department of Computer Science, University of Colorado, Boulder

This is an extended abstract that overviews the main results of the full versions of two separate but related manuscripts that will be published elsewhere. This research has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 802020-ERC-HARMONIC.

Abstract

A family $\mathcal{F} \subseteq \binom{[n]}{k}$ of k-sets of an n-element set $[n] = \{1, 2, \cdots, n\}$ is *t-intersecting* if $|x \cap y| \geq t$ for all $x, y \in \mathcal{F}$. In 1961, Erdős, Ko, and Rado showed for all $t \leq k < n/2$ that the largest t-intersecting families have size no greater than $\binom{n-t}{k-t}$, and moreover, that the extremal families are precisely the *canonically t-intersecting families* $\mathcal{F}_T$, i.e., the families obtained by taking all k-sets containing a given t-set $T \subseteq [n]$. This seminal result has since been generalized to a variety of other combinatorial domains collectively known as *Erdős–Ko–Rado combinatorics*. In this area, spectral techniques have been quite effective for bounding the sizes of t-intersecting families of various combinatorial objects, but general methods for *characterizing* the extremal t-intersecting families for all n, t remain elusive.

We present new spectral techniques for characterizing the extremal t-intersecting families of various combinatorial domains for small t. We use these techniques to prove a couple of Erdős–Ko–Rado conjectures on the characterization of extremal 2-intersecting families, namely, for 2-intersecting families of permutations S_n, 2-intersecting families of perfect matchings of K_{2n}, and so-called *partially 2-intersecting families* of perfect matchings of the complete k-uniform hypergraph K^k_{kn} on kn vertices for k fixed and n sufficiently large.

1 Introduction

Let $\binom{[n]}{k}$ be the collection of k-sets of an n-element set $[n] := \{1, 2, \cdots, n\}$. A family $\mathcal{F} \subseteq \binom{[n]}{k}$ is *t-intersecting* if $|S \cap T| \geq t$ for all $S, T \in \mathcal{F}$. We say that a family $\mathcal{F}$ is *canonically t-intersecting* if there exists a t-set $T \subseteq [n]$ such that $\mathcal{F} = \mathcal{F}_T := \{S \in \binom{[n]}{k} : T \subseteq S\}$. In [5], Erdős, Ko, and Rado characterized the largest t-intersecting families for sufficiently large n.

Theorem 1 (Erdős–Ko–Rado '61)**.** *For all $n \geq t + (k-t)\binom{k}{t}^3$, if $\mathcal{F} \subseteq \binom{[n]}{k}$ is t-intersecting, then*

$$|\mathcal{F}| \leq \binom{n-t}{k-t}.$$

Moreover, equality holds if and only if $\mathcal{F}$ is a canonically t-intersecting family.

[1]Email: giladchase@cs.technion.ac.il.
[2]Email: netad@cs.technion.ac.il.
[3]Email: yuvalfi@cs.technion.ac.il. Research of Y. F. supported by the European Union's Horizon 2020 research and innovation programme under grant agreement No 802020-ERC-HARMONIC.
[4]Email: nathan.lindzey@colorado.edu.

The dependency on n was subsequently improved by Wilson to hold for all $n \geq (t+1)(k-t-1)$, which is best possible. Many different proofs of the Erdős–Ko–Rado theorem have since appeared and it has been extended to a variety of combinatorial domains. These results are collectively known as *Erdős-Ko-Rado (EKR) combinatorics*. Algebraic/spectral techniques have played a distinguished role in EKR combinatorics, so much so that a textbook has been written on the subject [8]. We overview the modus operandi of such proofs.

1.1 Spectral Methods

Typically one starts by constructing a graph $\Gamma = (\mathcal{X}, E)$ on a domain of objects $\mathcal{X}$ defined such that $xy \in E$ if x and y do not t-intersect (for some suitably defined notion of intersection). Recall that a set $S \subseteq \mathcal{X}$ is *independent* if $xy \notin E$ for all $x, y \in S$. By design, the independent sets of Γ are t-intersecting families; therefore, to deduce an EKR result for t-intersecting families of $\mathcal{X}$, it suffices to characterize the maximum independent sets of Γ. At first it is not clear how useful this graphical reformulation is until one considers the Delsarte–Hoffman *ratio bound*, stated below. Recall that a *pseudo-adjacency matrix* A of a regular graph $G = (V, E)$ is a $|V| \times |V|$ matrix with constant row sum such that $ij \notin E \Rightarrow A_{ij} = 0$.

Theorem 2 (Ratio Bound)**.** *Let A be a pseudo-adjacency matrix of a regular graph $G = (V, E)$ with eigenvalues $\lambda_{\max} \geq \cdots \geq \lambda_{\min}$. If $S \subseteq V$ is independent, then*

$$|S| \leq |V| \frac{-\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}}.$$

If equality holds, then the characteristic vector $1_S \in \mathbb{R}^V$ of S lies in the span of the greatest and least eigenspace of A.

Let Σ be the universe of atoms that compose the members of $\mathcal{X}$ so that $x \subseteq \Sigma$ for all $x \in \mathcal{X}$. Big t-intersecting families $\mathcal{F}_T \subseteq \mathcal{X}$ can be constructed by taking all $x \in \mathcal{X}$ that contain some fixed t-set of atoms $T \subseteq \Sigma$ so that any two members of the family contain T in their intersection. These are the so-called *canonically t-intersecting families*, also known as *trivially t-intersecting families* or *t-stars*. Remarkably, it is often the case that Γ admits a pseudo-adjacency matrix with eigenvalues such that the ratio bound equals the size of a canonically t-intersecting family, thus proving that the canonically t-intersecting families are indeed maximum intersecting families.

This is a powerful technique for proving the *bound* of an EKR result, but often the most difficult step of these spectral methods is leveraging the spectral consequences of the case of equality in the ratio bound to *characterize* the extremal families, i.e., prove that the canonically t-intersecting families are the *only* maximum intersecting families (see [8, pg. 22], for example).

1.2 Characterizing Extremal t-intersecting Families

By now there are many different methods for characterizing the extremal 1-intersecting families of various combinatorial domains (see [8] for an overview of these techniques). In the full version [7], we give a simple spectral method for characterizing extremal 1-intersecting families. It is based off the trivial observation that there is always a member of an intersecting family that can be perturbed to lie outside the family, and that one can often choose this perturbation to be sufficiently local for combinatorial objects. This general principle underlies recent work on *stability* versions of asymptotic EKR theorems (see [4, 2], for example), but here we show that this principle can in fact lead to succinct proofs of the characterization of the extremal 1-intersecting families in the *exact* regime, i.e., for all n. We sketch this technique in Section 4 and use it to prove a conjecture on the characterization of the largest *partially* 2-intersecting families of perfect hypermatchings of the complete k-uniform hypergraph on kn vertices (see Section 4 for definitions and a sketch of the proof).

Theorem 3 (Filmus, Lindzey '22)**.** [12, Conjecture 6.1] *If $\mathcal{F} \subseteq \mathcal{M}_{kn}^k$ is a largest partially 2-intersecting family, then $\mathcal{F}$ is a canonically partially 2-intersecting family provided that n is sufficiently large.*

In [7] we also use this method to give simpler and shorter proofs of many well-known EKR theorems.

Unfortunately, the known methods for characterizing extremal 1-intersecting families only generalize to arbitrary t in just a handful of cases (e.g., subsets, subspaces, cometric association schemes). In particular, these techniques all seem to fall short for more complicated combinatorial objects such as permutations and perfect matchings, and many challenging open questions remain in this area.[5]

We give a new method for characterizing the extremal t-intersecting families of combinatorial domains for small t that we use to prove the following (see Sections 2 and 3 for definitions).

Theorem 4 (Chase, Dafni, Filmus, Lindzey '22)**.** *Let $n \geq 2$. If $\mathcal{F} \subseteq S_n$ is a largest 2-intersecting family, then $\mathcal{F}$ is a canonically 2-intersecting family.*

Theorem 5 (Chase, Dafni, Filmus, Lindzey '22)**.** *Let $n \geq 2$. If $\mathcal{F} \subseteq \mathcal{M}_{2n}$ is a largest 2-intersecting family, then $\mathcal{F}$ is a canonically 2-intersecting family.*

Our method approaches these problems from a theoretical computer science vantage point, namely, *the analysis of Boolean functions* [13]. This area has traditionally focused on the hypercube $\mathbb{Z}_2^n$, but recently this theory has been extended to a broader variety of combinatorial domains that includes permutations and perfect matchings. Central to this theory is the notion of a *complexity measure* of a Boolean function, which roughly speaking measures how "complex" a Boolean function is with respect to a certain property (e.g., polynomial degree, Fourier-degree complexity, sensitivity, see [13]).

Here, the Boolean functions (0/1-functions) that we consider are the characteristic functions of the canonically t-intersecting families of permutations and perfect matchings. A recurring phenomenon in EKR combinatorics is that these functions have Fourier-degree t and that they span the space of Fourier-degree t functions. A fact established in [1] that underlies our method is that Boolean functions f of Fourier-degree t must have low *certificate complexity*, i.e., $C(f) = O(\text{poly}(t))$ (see Section 2.1), which in some sense measures the Boolean function's "combinatorial complexity".[6] However, the results of [1] are not sharp enough to deduce the characterization of the extremal 2-intersecting families of permutations and perfect matchings for all n. Thus we require a more refined approach that we sketch in Sections 2 and 3

The remainder of this paper is outlined as follows. In Section 2 we review the EKR theorem 2-intersecting permutations and sketch its proof. A similar proof holds for the EKR theorem for 2-intersecting perfect matchings which we cover in Section 3. Finally, in Section 4 we describe a different proof technique used to prove the conjecture on partially 2-intersecting families (Theorem 3).

2 2-Intersecting Families of Permutations

Let S_n be the group of all permutations of $[n]$. Two permutations $\alpha, \beta \in S_n$ are *t-intersecting* if there exist t distinct indices $i_1, \ldots, i_t \in [n]$ such that $\alpha(i_1) = \beta(i_1), \ldots, \alpha(i_t) = \beta(i_t)$. Ellis, Friedgut and Pilpel [3] conjectured the following.

Conjecture 6. Let $\mathcal{F}$ be a subset of S_n which is t-intersecting. Then

$$|\mathcal{F}| \leq \max_{r \leq (n-t)/2} |\{\alpha \in S_n : \alpha(i) = i \text{ for at least } t + r \text{ many } i \in [t + 2r]\}| .$$

Furthermore, if $\mathcal{F}$ achieves this bound, then

$$\mathcal{F} = \{\alpha \in S_n : \alpha(i_s) = j_s \text{ for at least } t + r \text{ many } s \in [t + 2r]\}$$

for some $r \leq (n - t)/2$ and some distinct $i_1, \ldots, i_{t+2r} \in [n]$ and distinct $j_1, \ldots, j_{t+2r} \in [n]$.

[5] We note that there has been some success characterizing extremal t-intersecting families in the *asymptotic regime*, where one allows n to be sufficiently large with respect to t; however, in this work we are interested in the more difficult task of deriving *exact* results that hold for all n, t.

[6] Roughly speaking, this means that 1-valued (or 0-valued) elements of the function f are all combinatorially quite similar, so we only need to perform a few tests to certify whether an element is 1-valued or not.

When $t \leq 3$, the maximum is always attained at $r = 0$, in which case the upper bound is $(n-t)!$, and the conjectured extremal families are the canonically t-intersecting families:

$$\mathcal{F}_{I \mapsto J} := \{\sigma \in S_n : \sigma(i_k) = j_k \text{ for all } 1 \leq k \leq t\} \quad I = (i_1, \cdots, i_t),\ J = (j_1, \cdots, j_t) \text{ distinct.}$$

Deza and Frankl proved the bound of 6 in the special case $t = 1$ by noticing that the n cyclic rotations of any fixed permutation are pairwise non-1-intersecting. Characterizing the extremal families in this case proved a lot harder, but eventually many proofs were found (see [8]).

The case $t > 1$ is significantly harder. For every $t > 1$, Ellis, Friedgut and Pilpel [3] proved that the upper bound in 6 holds for large enough n, and Ellis [2] proved that the characterization of 6 holds for large enough n. Recently, Meagher and Razafimahatratra [11], refining the techniques of Ellis, Friedgut and Pilpel, proved that 2-intersecting families of S_n contain at most $(n-2)!$ permutations, thus verifying the upper bound part of 6 for $t = 2$ and *all* n. Their proof also shows that the extremal 2-intersecting families have Fourier-degree 2. This implies through known results [3] that extremal 2-intersecting families have polynomial degree 2, which we refer to henceforth as simply *degree* (see [1] for a more detailed discussion of these complexity measures). However, Meagher and Razafimahatratra were unable to characterize the extremal families, and they conjectured that the canonically 2-intersecting families are the only extremal families. We prove this conjecture for all n.

Theorem 4 (restated). *Let $n \geq 2$. If $\mathcal{F} \subseteq S_n$ is a largest 2-intersecting family, then $\mathcal{F}$ is a canonically 2-intersecting family.*

The main idea in our proof of this result is to work with the *certificate complexity* of extremal 2-intersecting families, which we formally define in the next section.

2.1 Certificate Complexity and the Proof of Theorem 4

Let $f\colon S_n \to \{0,1\}$, and let $\alpha \in S_n$ be such that $f(\alpha) = b$. A *certificate* for α is a subset $\{i_1, \ldots, i_m\} \subseteq [n]$ such that $f(\beta) = b \in \{0,1\}$ whenever $\beta(i_1) = \alpha(i_1), \ldots, \beta(i_m) = \alpha(i_m)$. The idea is that in order to verify that $f(\alpha) = b$, it suffices to check the value of $\alpha(i_1), \ldots, \alpha(i_m)$. The *certificate complexity* of α is the minimum size of a certificate for α. The certificate complexity of f is the maximum, over all $\alpha \in S_n$, of the certificate complexity of α. We denote the certificate complexity of f by $C(f)$.

We are now ready to sketch the proof of Theorem 4. First, a simple argument, using another complexity measure called *sensitivity* (see [1]), shows that if $n \geq 8$ and $f\colon S_n \to \{0,1\}$ has degree at most 2 (which is true of any extremal 2-intersecting family), then $C(f) \leq n-2$.

Proposition 7. *If $n \geq 8$ and $f : S_n \to \{0,1\}$ has degree at most 2, then $C(f) \leq n-2$.*

Our key lemma is a structural result for 2-intersecting families f of degree at most 2 and $C(f) \leq n-2$.

Lemma 8 (Key Lemma)**.** *Let $\mathcal{F} \subseteq S_n$ be a 2-intersecting family whose characteristic function f has degree at most 2, and suppose that $C(f) \leq n-2$. Then $\mathcal{F}$ is contained in the union of at most T many canonically $C(f)$-intersecting families, where $T = \frac{2\lfloor C(f)/2 \rfloor (C(f)-1)!}{2^{\lfloor C(f)/2 \rfloor}}$. Moreover, $C(f) \geq 2$.*

We then use the key lemma to prove that the size of the 2-intersecting family must be bounded away from $(n-2)!$ unless $C(f) = 2$. But if $C(f) = 2$, then by definition, f is a characteristic function of a canonically 2-intersecting family, which completes the proof provided that $n \geq 8$. Fortunately, the remaining cases $2 \leq n \leq 8$ are small enough to be brute-forced to complete the proof for all $n \geq 2$.

3 2-Intersecting Families of Perfect Matchings of K_{2n}

Let $\mathcal{M}_{2n}$ consist of all perfect matchings of K_{2n}. Two perfect matchings in $\mathcal{M}_{2n}$ are *t-intersecting* if they have at least t edges in common. The canonically t-intersecting families of $\mathcal{M}_{2n}$ are of the form $\mathcal{F}_T := \{m \in \mathcal{M}_{2n} : T \subseteq m\}$ for any t-set T of disjoint edges of K_{2n}.

A simple argument shows that a 1-intersecting family of perfect matchings contains at most $(2n-3)!! = (2n-3)(2n-5)\cdots(1)$ perfect matchings. More difficult arguments (see [8]) show that this is attained uniquely by canonically 1-intersecting families. Lindzey [10, 9] extended the arguments of Ellis, Friedgut and Pilpel [3, 2] to the setting of perfect matchings, proving that for all t, the maximum size t-intersecting families are the canonically t-intersecting families, for large enough n. Recently, Fallat, Meagher and Shirazi [6] showed that the maximum size of a 2-intersecting family is at most $(2n-5)!!$ for all n, but they were unable to characterize the extremal 2-intersecting families. We prove an analog of Theorem 4 in the setting of perfect matchings.

Theorem 5 (restated). *Let $n \geq 2$. If $\mathcal{F} \subseteq \mathcal{M}_{2n}$ is a largest 2-intersecting family, then $\mathcal{F}$ is a canonically 2-intersecting family.*

An analogous certificate complexity-based proof works in the case of the perfect matchings, but with one complication. Our proof of Lemma 8 in the permutation setting depends on the well-known fact that Boolean degree-1 functions on S_n have certificate complexity 1. This is no longer the case for Boolean degree-1 functions on $\mathcal{M}_{2n}$ (see [1]), but nevertheless we are able to overcome this obstacle and derive an analogous key lemma.

4 Partially 2-Intersecting Families of Perfect Hypermatchings of K_{kn}^k

Let $\mathcal{M}_{kn}^k$ be the set of perfect hypermatchings of K_{kn}^k. Two perfect hypermatchings $m, m' \in \mathcal{M}_{kn}^k$ are said to *partially 2-intersect* if there exists a pair of k-edges $e \in m$, $e' \in m'$ such that $|e \cap e'| \geq 2$. Meagher et al. [12] show for sufficiently large n that if $\mathcal{F} \subseteq \mathcal{M}_{kn}^k$ is a partially 2-intersecting family, then $|\mathcal{F}| \leq \binom{kn-2}{k-2}|\mathcal{M}_{k(n-1)}^k|$. The canonically partially 2-intersecting families $\mathcal{F}_{ij} = \{m \in \mathcal{M}_{kn}^k : ij \subseteq e \in m\}$ where $ij \subseteq [kn]$ meet this bound with equality. Their proof uses the ratio bound, and since equality is met, a consequence of their proof is that any extremal partially 2-intersecting family lives in the direct sum of the greatest eigenspace V_0 and least eigenspace V_1 of the *partial 2-derangement graph* $\Gamma = (\mathcal{M}_{kn}^k, E)$ defined such that m, m' are adjacent if there exists no pair of edges $e \in m$, $e' \in m'$ such that $|e \cap e'| \geq 2$. Meagher et al. conjectured for sufficiently large n that the canonically partially 2-intersecting families are the only extremal partially 2-intersecting families. The main result of this section is Theorem 3 which proves their conjecture.

Theorem 3. *If $\mathcal{F} \subseteq \mathcal{M}_{kn}^k$ is a largest partially 2-intersecting family, then $\mathcal{F}$ is a canonically partially 2-intersecting family provided that n is sufficiently large.*

The proof of this result is a bit more involved than the proofs of the previous results, as the representation theory of $\mathcal{M}_{kn}^k$ is far from understood, which prohibits us from using the complexity measure framework of [1] for perfect hypermatchings.[7] Instead, we prove Theorem 3 using a simple spectral approach sketched as follows.

First, we use representation-theoretical techniques to derive closed forms for the orthogonal projector P onto the direct sum $V_0 \oplus V_1$. In particular, we give a combinatorial expression for the components of the projection Pf where f is the characteristic function of any family $\mathcal{F} \subseteq \mathcal{M}_{kn}^k$. The combinatorial expression is linear in the sizes of *ij-restrictions* $\mathcal{F}\downarrow_{ij} := \{m \in \mathcal{F} : ij \in e \text{ for some } e \in m\}$ for any ij. Results of [12] show that $Pf = f$ for any extremal partially 2-intersecting family $\mathcal{F}$, which provides us with a combinatorial expression for the image of f.

Second, we pick two perfect hypermatchings $m_1 \in \mathcal{F}$ and $m_0 \notin \mathcal{F}$ so that m_1 and m_0 share as many hyperedges as possible. Using our closed combinatorial formula for the projection Pf and the fact that $Pf = f$ allows us to establish the following relation:

$$f(m_1) - f(m_0) = \left(\frac{(2n+k-3)(kn-3)}{2(k-1)(n-1)|\mathcal{M}_{kn}^k|}\right)\left[\sum_{ij \subseteq e \in m_1} |\mathcal{F}\downarrow_{ij}| - \sum_{ij \subseteq e \in m_0} |\mathcal{F}\downarrow_{ij}|\right] = 1 - 0.$$

[7] For experts in representation theory, the set $\mathcal{M}_{kn}^k$ as a homogeneous space under the action of S_{kn} is not a Gelfand pair and the dimensions of its isotypic components are unknown.

Since we chose m_1 and m_0 to be as close as possible, most of the bracketed terms cancel, as they share many hyperedges. This allows us to show there must be a large ij-restriction, in particular,

$$|\mathcal{F}\downarrow_{ij}| \geq \left(\frac{(n-1)}{(2n+k-3)(kn-3)}\right)|\mathcal{M}^k_{kn}| = \Omega\left(\binom{kn-2}{k-2}|\mathcal{M}^k_{k(n-1)}|\right).$$

Lastly, we use the existence of this large ij-restriction along with some combinatorial and isoperimetric arguments to show that $\mathcal{F}$ must in fact be the canonically partially 2-intersecting family $\mathcal{F}_{ij}$. We note that the requirement that n be sufficiently large stems from the proof method of [12], and that our techniques can likely be refined to hold for all n provided that the bound holds for all n.

References

[1] N. Dafni, Y. Filmus, N. Lifshitz, N. Lindzey, and M. Vinyals. "Complexity measures on the symmetric group and beyond". In: *12th Innovations in Theoretical Computer Science Conference*. Vol. 185. LIPIcs. Leibniz Int. Proc. Inform. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2021, Art. No. 87, 5. DOI: 10.1305/ndjfl/1040046140.

[2] D. Ellis. "Stability for t-intersecting families of permutations". In: *J. Combin. Theory Ser. A* 118.1 (2011), pp. 208–227. DOI: 10.1016/j.jcta.2010.04.005.

[3] D. Ellis, E. Friedgut, and H. Pilpel. "Intersecting families of permutations". In: *J. Amer. Math. Soc.* 24.3 (2011), pp. 649–682. DOI: 10.1090/S0894-0347-2011-00690-5.

[4] D. Ellis, N. Keller, and N. Lifshitz. "Stability versions of Erdős-Ko-Rado type theorems via isoperimetry". In: *J. Eur. Math. Soc. (JEMS)* 21.12 (2019), pp. 3857–3902. DOI: 10.4171/JEMS/915.

[5] P. Erdős, C. Ko, and R. Rado. "Intersection theorems for systems of finite sets". In: *Quart. J. Math. Oxford Ser. (2)* 12 (1961), pp. 313–320. DOI: 10.1093/qmath/12.1.313.

[6] S. Fallat, K. Meagher, and M. N. Shirazi. "The Erdős-Ko-Rado theorem for 2-intersecting families of perfect matchings". In: *Algebr. Comb.* 4.4 (2021), pp. 575–598. DOI: 10.5802/alco.169.

[7] Y. Filmus and N. Lindzey. *Simple Algebraic Proofs of Uniqueness for Erdős-Ko-Rado Theorems*. Preprint. 2022. DOI: 10.48550/ARXIV.2201.02887.

[8] C. Godsil and K. Meagher. *Erdős-Ko-Rado theorems: algebraic approaches*. Vol. 149. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2016, pp. xvi+335. DOI: 10.1017/CBO9781316414958.

[9] N. Lindzey. "Matchings and representation theory". PhD thesis. University of Waterloo, 2018.

[10] N. Lindzey. "Stability for Intersecting Families of Perfect Matchings". In: (2018). Preprint. DOI: 10.48550/ARXIV.1808.03453.

[11] K. Meagher and A. S. Razafimahatratra. "The Erdős-Ko-Rado theorem for 2-pointwise and 2-setwise intersecting permutations". In: *Electron. J. Combin.* 28.4 (2021), Paper No. 4.10, 21. DOI: 10.37236/9556.

[12] K. Meagher, M. N. Shirazi, and B. Stevens. *An Extension of the Erdős-Ko-Rado Theorem to uniform set partitions*. Preprint. 2021. DOI: 10.48550/ARXIV.2108.07692.

[13] R. O'Donnell. *Analysis of Boolean functions*. Cambridge University Press, New York, 2014, pp. xx+423. DOI: 10.1017/CBO9781139814782.

Combinatorics and preservation of conically stable polynomials (Extended abstract)

Giulia Codenotti[1], Stephan Gardoll[2], Thorsten Theobald[3]

Goethe-Universität, FB 12 – Institut für Mathematik, Postfach 11 19 32, 60054 Frankfurt am Main

The full version of this work will be posted on the arXiv and will be published elsewhere. This research is supported by DFG grant TH 1333/7-1.

1 Introduction

Multivariate stable polynomials can be seen as a generalization of real-rooted polynomials and they enjoy striking connections to combinatorics (see, for example, [8]). A groundbreaking result of Choe, Oxley, Sokal and Wagner states that the support of a multi-affine, homogeneous and stable polynomial $f \in \mathbb{R}[\mathbf{z}] = \mathbb{R}[z_1, \ldots, z_n]$ is the set of bases of a matroid [4]. Brändén [2] proved a generalization for the support of any stable polynomial $f \in \mathbb{R}[\mathbf{z}]$, showing that it forms a jump system, i.e., it satisfies the so-called two-step axiom. See Section 2 for formal definitions. Recently, Rincón, Vinzant and Yu gave an alternative proof of the matroid result, based on a tropical proof of the auxiliary statement that positive hyperbolicity of a variety is preserved under passing over to the initial form [9].

The proofs of these combinatorial properties strongly rely on *preservation properties* of stable polynomials. These preservation properties establish the connection between the combinatorial and the algebraic viewpoint. For example, taking the partial derivative of a polynomial f shifts the support vectors of f by a unit vector in a negative coordinate direction (and some support vectors may disappear). Since stability of a polynomial is preserved under taking partial derivatives, one can use this preserver to argue about the combinatorics of the support. In the univariate case, these considerations are classical for deriving log-concavity of sequences with real-rooted generating functions.

Stable polynomials are closely related to *hyperbolic polynomials* [6]. Moreover, further variants and generalizations have been developed, including *conic stability* introduced by Jörgens and the third author [7], *Lorentzian polynomials* introduced by Brändén and Huh [3] and *positively hyperbolic varieties* introduced by Rincón, Vinzant and Yu [9].

The purpose of the current work is to initiate the study of the generalization of the combinatorics of stable polynomials towards conically stable polynomials as well as the underlying preservation operators needed for the proofs. A focus is on the transition from the classical stability (which coincides with conic stability w.r.t. the non-negative orthant) to conic stability w.r.t. the positive semidefinite cone.

For proofs, we refer to the full version of the work.

2 Preliminaries

Let $\mathbb{R}_+$ and $\mathbb{R}_{>0}$ denote the sets of non-negative and of positive real numbers. Further, let $\mathcal{H} := \{z \in \mathbb{C} : \Im(z) > 0\}$ be the *open upper half-plane* of $\mathbb{C}$, where $\Im$ denotes the imaginary part of a complex number. For a vector $\mathbf{z} \in \mathbb{C}^n$, $\Im(\mathbf{z})$ denotes the vector of the imaginary parts of the components of $\mathbf{z}$.

[1]Email: codenott@math.uni-frankfurt.de. G. C. was also supported by an Early Career Researcher grant of Goethe University.

[2]Email: gardoll@math.uni-frankfurt.de.

[3]Email: theobald@math.uni-frankfurt.de.

2.1 Stable polynomials

A polynomial $f \in \mathbb{C}[\mathbf{z}]$ is called *stable* if for every root $\mathbf{z}$ of f there exists some $j \in [n]$ with $\Im(z_j) \leq 0$. Hence, a univariate real polynomial f is stable if and only if it is real-rooted, because the non-real roots of univariate real polynomials occur in conjugate pairs. The following operations preserve stability of a polynomial $f \in \mathbb{C}[\mathbf{z}]$, where the second property can be derived from the Gauß-Lucas Theorem.

1. *Inversion*: $z_1^{\deg_1(f)} \cdot f(-z_1^{-1}, z_2, \ldots, z_n)$ is stable, where $\deg_i$ is the degree in the variable z_i.
2. *Differentiation*: $\partial_j f(\mathbf{z})$ is stable for every $j \in [n]$.

The stability preservation under inversion is a key ingredient in the jump system result of Theorem 2. A prominent linear stability preserver is the Lieb-Sokal Lemma (see, e.g., [10]). It is an essential ingredient in Borcea and Brändén's full characterization of linear operators preserving stability.

Proposition 1 (Lieb-Sokal Lemma)**.** *Let $g(\mathbf{z}) + yf(\mathbf{z}) \in \mathbb{C}[\mathbf{z}, y]$ be stable and assume $\deg_i(f) \leq 1$ for some $i \in [n]$. Then $g(\mathbf{z}) - \partial_i f(\mathbf{z}) \in \mathbb{C}[\mathbf{z}]$ is stable or identically zero.*

The initial form $\mathrm{in}_w(f)$ of a polynomial $f(x) = \sum_{\alpha \in S} c_\alpha x^\alpha$ w.r.t. a functional $w \in (\mathbb{R}^n)^*$ is defined as $\mathrm{in}_w(f) = \sum_{\alpha \in S_w} c_\alpha x^\alpha$, where $S_w := \{\alpha \in S : \langle w, \alpha \rangle = \max_{\beta \in S} \langle w, \beta \rangle\}$. That is, we restrict the polynomial f to those monomials whose exponents lie on the face of the Newton polytope of f where the functional w is maximized. Using methods from tropical geometry, Rincón, Vinzant and Yu [9] showed that for polynomials with real coefficients, stability is preserved under taking initial forms.

2.2 Combinatorics of stable polynomials

For $\alpha, \beta \in \mathbb{Z}^n$, the *steps* between α and β are defined as the set

$$\mathrm{St}(\alpha, \beta) := \{\sigma \in \mathbb{Z}^n : |\sigma| = 1, |\alpha + \sigma - \beta| = |\alpha - \beta| - 1\},$$

where $|\sigma| := \sum_{i=1}^n \sigma_i$. A collection of points $\mathcal{F} \subseteq \mathbb{Z}^n$ is called a *jump system* if for every $\alpha, \beta \in \mathcal{F}$ and $\sigma \in \mathrm{St}(\alpha, \beta)$ with $\alpha + \sigma \notin \mathcal{F}$ there is some $\tau \in \mathrm{St}(\alpha + \sigma, \beta)$ such that $\alpha + \sigma + \tau \in \mathcal{F}$. In words, if after one step from α towards β we have left the set $\mathcal{F}$, then there must be a second step that takes us back into $\mathcal{F}$. This property is also known as the *two-step axiom*. The following theorem reveals the connection between stable polynomials and jump systems.

Theorem 2 (Brändén [2])**.** *If $f \in \mathbb{C}[\mathbf{z}]$ is stable, then the support $\mathrm{supp}(f)$ is a jump system.*

For the special case of binomials, the following more detailed classification is known.

Theorem 3 (Rincón, Vinzant, Yu [9])**.** *Let $f = c_\alpha \mathbf{z}^\alpha + c_\beta \mathbf{z}^\beta$ with $c_\alpha, c_\beta \neq 0$ and $\alpha, \beta \in \mathbb{Z}_+^n$ be stable and let $\mathbf{z}^\alpha$ and $\mathbf{z}^\beta$ do not have a common factor. Then one of the following holds.*

1. *$\{\alpha, \beta\} = \{0, \mathbf{e}_i\}$ for some $i \in [n]$, where $\mathbf{e}_i$ denotes the i-th unit vector in $\mathbb{R}^n$.*
2. *$\{\alpha, \beta\} = \{\mathbf{e}_i, \mathbf{e}_j\}$ for some $i, j \in [n]$ and $\frac{c_\alpha}{c_\beta} \in \mathbb{R}_+$, or*
3. *$\{\alpha, \beta\} = \{0, \mathbf{e}_i + \mathbf{e}_j\}$ for some $i, j \in [n]$ and $\frac{c_\alpha}{c_\beta} \in \mathbb{R}_-$.*

2.3 Conic stability

The following notion of conic stability generalizes stability to more general cones. Let K be a closed, convex cone in $\mathbb{R}^n$ and denote by $\operatorname{relint} K$ its relative interior. A polynomial $f \in \mathbb{C}[\mathbf{z}]$ is called *K-stable*, if $f(\mathbf{z}) \neq 0$ whenever $\Im(\mathbf{z}) \in \operatorname{relint} K$. Observe that by choosing the cone $K = \mathbb{R}_+^n$, we recover the usual notion of stability. In the case of a homogeneous polynomial, K-stability of f is equivalent to the containment of $\operatorname{relint} K$ in a hyperbolicity cone of f. Conic stability can be characterized through stability of univariate polynomials.

Proposition 4 ([7]). *A polynomial $f \in \mathbb{C}[\mathbf{z}] \setminus \{0\}$ is K-stable if and only if for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $\mathbf{y} \in \operatorname{relint} K$, the univariate polynomial $t \mapsto f(\mathbf{x} + t\mathbf{y})$ is stable or identically zero.*

For simplicial cones $K = \operatorname{pos}\{\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}\}$ with linearly independent vectors $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}$, the notion of K-stability just differs from usual stability by a variable transformation.

2.4 Positive semidefinite stability

Stability with respect to the positive semidefinite cone on the space of symmetric matrices is denoted as *psd-stability* (in the homogeneous case such polynomials are also known as Dirichlet-Gårding polynomials). Prominent subclasses of psd-stable polynomials arise from determinantal representations [5]. Blekherman, Kummer, Sanyal et al. [1] have constructed a family of psd-stable *lpm-polynomials* (*linear principle minor polynomials*) from homogeneous multiaffine stable polynomials.

Formally, denote by $\mathcal{S}_n^{\mathbb{C}}$ the vector space of complex symmetric matrices and by $\mathcal{S}_n$ the space of real ones. The cones of real positive semidefinite and positive definite matrices are denoted by $\mathcal{S}_n^{+}$ and $\mathcal{S}_n^{++}$. Let $\mathbb{C}[Z]$ denote the ring of polynomials on the symmetric matrix variables $Z = (z_{ij})$. Psd-stability is defined as $\mathcal{S}_n^{+}$-stability for polynomials over the vector space $\mathcal{S}_n^{\mathbb{C}}$.

Monomials in $\mathbb{C}[Z]$ can be written as $Z^{\alpha} = \prod_{1 \le i,j \le n} z_{ij}^{\alpha_{ij}}$ with some non-negative symmetric matrix α whose diagonal entries are integers and whose off-diagonal entries are half-integers. The *support* $\operatorname{supp}(f)$ of such a polynomial is then the collection of symmetric exponent matrices of the monomials occurring in the polynomial. The variables z_{ii} are called *diagonal variables*, while the variables z_{ij} with $i \neq j$ are the *off-diagonal variables*. We say that a monomial with exponent matrix α is a *diagonal monomial* if $\alpha_{ij} = 0$ for all $i \neq j \in [n]$, and we say that it is an *off-diagonal* monomial if $\alpha_{ii} = 0$ for all $i \in [n]$. By convention, we say that a constant is a diagonal monomial, but not an off-diagonal one.

Example 5. Let $f(Z) = \det(Z)$ in the polynomial ring $\mathbb{C}[Z]$ over the vector space $\mathcal{S}_2^{\mathbb{C}}$. Then

$$f(Z) = z_{11}z_{22} - z_{12}^2, \quad \text{or equivalently} \quad f(Z) = Z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - Z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The monomial $z_{11}z_{22}$ is a diagonal monomial while the other is an an off-diagonal monomial.

A prime example of psd-stable polynomials are determinants. We recall the short proof.

Lemma 6. *$f(Z) = \det(Z)$ is psd-stable.*

Proof. Suppose that f is not psd-stable, that is, there exist real symmetric matrices A and B with B positive definite, such that $f(A + iB) = 0$. Then B is invertible and $0 = f(A + iB) = \det(A + iB) = \det(B)\det(B^{-\frac{1}{2}}AB^{-\frac{1}{2}} + iI_n)$. Hence, $-i$ is a root of the characteristic polynomial of, and thus an eigenvalue of, $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$: a contradiction, since a symmetric real matrix has only real eigenvalues. □

Contrary to the usual stability notion, monomials are not necessarily psd-stable. In fact, every monomial with an off-diagonal variable as a factor, is not psd-stable since it evaluates to zero for $Z = i \cdot I_n$.

Psd-stability can be viewed as stability with respect to the *Siegel upper half-space* $\mathcal{H}_{\mathcal{S}} = \{A \in \mathbb{C}^{n \times n} \text{ symmetric } : \Im(A) \text{ is positive definite}\}$. The Siegel upper half-space occurs in algebraic geometry and number theory as the domain of modular forms.

3 Preservers for conic stability

We generalize several preserving operators for usual stability to the conic stability w.r.t. some closed, convex cone K. We focus here on the preservation of stability under taking directional derivatives and a conic version of the Lieb-Sokal Lemma. Specific preservers for psd-stability will be studied in the next section. For a vector $\mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, denote by $\partial_{\mathbf{v}}$ the directional derivative in direction $\mathbf{v}$.

Lemma 7. *Let $f \in \mathbb{C}[\mathbf{z}]$ be K-stable. For $\mathbf{v} \in K$, the polynomial $\partial_{\mathbf{v}} f$ is K-stable or identically zero.*

In the homogeneous case, this statement follows from the concept of a Renegar derivative for hyperbolic polynomials. In the usual Lieb-Sokal Lemma 1, we take a partial derivative of a polynomial which has degree at most 1 in the corresponding variable. To formulate a generalized result for arbitrary cones, we need a generalized notion of degree. For $\mathbf{v} \in \mathbb{R}^n$, we call $\rho_{\mathbf{v}}(f)$ the *degree of f in direction* $\mathbf{v}$, defined as the degree of the univariate polynomial $f(\mathbf{w} + t\mathbf{v}) \in \mathbb{C}[t]$ for generic $\mathbf{w} \in \mathbb{C}^n$. The degree in the direction of a unit vector $\mathbf{e}^{(j)}$ coincides with the univariate degree with respect to the variable j. We can state the formulation of the conic Lieb-Sokal stability preservation.

Theorem 8 (Conic Lieb-Sokal stability preservation)**.** *Let K' be given by $K' = K \times \mathbb{R}_+$ and $g(\mathbf{z}) + yf(\mathbf{z}) \in \mathbb{C}[\mathbf{z}, y]$ be K'-stable and such that $\rho_{\mathbf{v}}(f) \leq 1$ for some $\mathbf{v} \in K$. Then $g - \partial_{\mathbf{v}} f$ is K-stable or $g - \partial_{\mathbf{v}} f \equiv 0$.*

4 Preservers for psd-stability

If $f \in \mathbb{C}[Z]$ is psd-stable, then, as a consequence of the preserver for conic stability, the differentiation $\partial_V f(Z)$ is psd-stable or zero for $V \in \mathcal{S}_n^+$. The diagonalization operation, which sets all off-diagonal variables of a psd-stable polynomial to zero, preserves psd-stability as well and connects to the usual stability notion. As the main result on preservers for psd-stability, we can show the preservation under a natural generalization inversion operator.

Theorem 9 (Psd-stability preservation under inversion)**.** *If $f(Z) \in \mathbb{C}[Z]$ is psd-stable, then the polynomial $\det(Z)^{\deg(f)} \cdot f(-Z^{-1})$ is psd-stable.*

Here, the factor $\det(Z)^{\deg(f)}$ serves to ensure that the product is a polynomial again. The proof generalizes the proof of preservation of usual stability under inversion, by using in addition various spectral properties of the relevant matrices. We also state a slightly more general version which resembles the existing formulation of the scalar version in Section 2.1.

Corollary 10. *If Z is a symmetric block diagonal matrix with blocks $Z_1, \ldots, Z_k$ and $f(Z) = f(Z_1, \ldots, Z_k)$ is psd-stable, then $\det(Z_1)^{\deg_{Z_1} f} \cdot f(-Z_1^{-1}, Z_2, \ldots, Z_k)$ is a psd-stable polynomial. Here, $\deg_{Z_1} f$ denotes the total degree of f with respect to the variables from the block Z_1.*

We close the section by providing a counterexample which shows that the stability preservation under taking the initial form does not generalize to the case of psd-stability.

Example 11. The polynomial $f \in \mathbb{C}[Z]$ given by $f(Z) = \det Z = z_{11}z_{22}z_{33} - z_{11}z_{23}^2 - z_{22}z_{13}^2 - z_{33}z_{12}^2 + 2z_{12}z_{13}z_{23}$ is a psd-stable polynomial. However, taking the initial form $\mathrm{in}_w(f)$ for $w = \frac{1}{4}\begin{pmatrix} 4 & 1 & 6 \\ 1 & 4 & 6 \\ 6 & 6 & 0 \end{pmatrix}$ yields $\mathrm{in}_w(f) = -z_{11}z_{23}^2 - z_{22}z_{13}^2 + 2z_{12}z_{13}z_{23}$, which vanishes at $Z = iI_3$. Since the imaginary part of iI_3 is a positive definite matrix, $\mathrm{in}_w(f)$ is not psd-stable.

5 Combinatorics of psd-stable polynomials

We study combinatorial properties of the support of psd-stable polynomials, inspired by the results on the support of stable polynomials. We prove a necessary condition on the support of any psd-stable polynomial in $\mathbb{C}[Z]$ and characterize the support of special families of psd-stable polynomials. In particular, we give a necessary criterion for psd-stability of a class of polynomials with arbitrary degree, characterize psd-stability of binomials and give a class of polynomials of determinants, which satisfies a generalized jump system criterion with regard to psd-stability.

Theorem 12. *If an off-diagonal variable z_{ij} (where $i < j$) occurs in a psd-stable polynomial $f \in \mathbb{C}[Z]$, then the corresponding diagonal variables z_{ii} and z_{jj} must also occur in f.*

This mimics the basic fact about positive semidefinite matrices that if an off-diagonal entry is non-zero, the corresponding diagonal entries must also be non-zero. For binomials we may extend Theorem 12 as follows.

Lemma 13. *Let $f(Z) = c_\alpha Z^\alpha + c_\beta Z^\beta$ be a psd-stable binomial. If the two monomials Z^α and Z^β do not have a common factor, then either both consist only of diagonal variables, or one only of diagonal and the other only of off-diagonal variables.*

5.1 Binomials and non-mixed polynomials

We give characterizations of the support and, as a consequence, of the structure of psd-stable binomials. We will state some of the results for the more general family of non-mixed polynomials. A polynomial $f \in \mathbb{C}[Z]$ is called *non-mixed* if every monomial that occurs in f either consists only of diagonal variables or only of off-diagonal variables. We always write such a non-mixed polynomial as $f = \sum_{\alpha \in A} c_\alpha Z^\alpha + \sum_{\beta \in B} c_\beta Z^\beta$, where A refers to the exponents of diagonal monomials and B refers to the exponents of off-diagonal monomials. It is useful to consider this larger family because it is closed under taking directional derivatives while binomials are not. The first criterion on the combinatorics of the support is that homogeneous non-mixed polynomials of high degree cannot be psd-stable.

Theorem 14. *Let $f(Z) = \sum_{\alpha \in A} c_\alpha Z^\alpha + \sum_{\beta \in B} c_\beta Z^\beta$ be a homogeneous non-mixed polynomial of degree $d \geq 3$ and assume $B \neq \emptyset$. Then f is not psd-stable.*

The second main theorem gives a complete characterization of the support of psd-stable binomials, analogous to the classification of stable binomials from Theorem 3.

Theorem 15. *Every psd-stable binomial is of one of the following forms.*

1. *Only diagonal variables appear in f and f satisfies the conditions of Theorem 3: $f(Z) = Z^\gamma(c_1 Z^{\alpha_1} + c_2 Z^{\alpha_2})$ with $|\alpha_1 - \alpha_2| \leq 2$ and at least one of α_1, α_2 is non-zero,*
2. *$f(Z) = Z^\gamma(c_1 z_{ii} z_{jj} + c_2 z_{ij}^2)$ with $i < j$,*

where $c_1, c_2 \neq 0$ and Z^γ is a diagonal monomial.

5.2 Polynomials of determinants

We show that the following class of polynomials of determinants satisfies a generalized jump system criterion with regard to psd-stability. Suppose that the symmetric matrix of variables Z is a diagonal block matrix with blocks $Z_1, \ldots, Z_k$. A *polynomial of determinants* is a polynomial in Z of the form $f(Z) = \sum_\alpha c_\alpha \det(Z)^\alpha$, where we define $\det(Z)^\alpha = \det(Z_1)^{\alpha_1} \cdots \det(Z_k)^{\alpha_k}$.

We say a polynomial of determinants $f(Z) = \sum_\alpha \det(Z)^\alpha$ is in its *standard form* if the largest possible determinantal monomial is factored out, i.e., $f(Z) = \det(Z)^\gamma \sum_\beta c_\beta \det(Z)^\beta = \det(Z)^\gamma \tilde{f}(Z)$, and all $c_\beta \neq 0$. Let $f(Z_1, \ldots, Z_k) = \sum_\alpha c_\alpha \det(Z)^\alpha$ be a polynomial of determinants. Then the *determinantal support* is defined as $\mathrm{supp}_{\det}(f) = \{\alpha \in \mathbb{Z}^k : c_\alpha \neq 0\}$. Note that the determinantal support specializes to the usual support when Z is a diagonal matrix, that is, all Z_i are 1×1 matrices of a single variable. As a corollary of Theorem 2, we obtain the following analogue for the determinantal support of psd-stable polynomials of determinants.

Corollary 16. *Let $f(Z_1, \ldots, Z_k) = \sum_\gamma c_\gamma \det(Z)^\gamma$ be psd-stable. Then the determinantal support of f forms a jump system.*

The next theorem shows that psd-stable polynomials of determinants have a very special structure.

Theorem 17. *Let* $f(Z) = \det(Z)^\gamma \sum_{\beta\in B} c_\beta \det(Z)^\beta = \det(Z)^\gamma \tilde{f}(Z)$ *be a psd-stable polynomial of determinants in standard form. Then any block* Z_i *appearing in* $\tilde{f}$ *(that is, any* Z_i *such that there is* $\beta \in B$ *with* $\beta_i > 0$*) has size* $k_i \leq 2$.

5.3 Outlook on the support of general psd-stable polynomials

By Theorem 2, the support of a stable polynomial defines a jump system. Hence, there cannot be large gaps in the support, that is, if two vectors are in the support and are far apart, there is some other vector of the support between them. The families studied in Section 5 suggest that a similar phenomenon happens for psd-stability: when there are too-large gaps in the support, the polynomial cannot be psd-stable.

In order to quantify what a *large gap* should be, we make two observations. First we note that restricting a psd-stable polynomial in the symmetric matrix variables Z to its diagonal yields a stable polynomial, thus between two monomials involving only diagonal variables the two-step axiom holds. A weaker statement is that between any two such monomials there is a sequence of linear and double steps which does not leave the support of the polynomial, where we define a *linear step* from a monomial to be multiplying the monomial by $z_{ij}^{\pm 1}$, a *double step* multiplying by $z_{ij}^{\pm 1} z_{kl}^{\pm 1}$.

Recall from Lemma 6 that a prominent example of psd-stable polynomials is the symmetric determinant $\det(Z)$. Its support has a special structure: it contains all monomials that can be obtained from $z_{11}, \ldots, z_{nn}$ by transpositions of indices, that is, by successively multiplying the monomial by $z_{ij} z_{kl} z_{ik}^{-1} z_{jl}^{-1}$ for some indices $i, j, k, l \in [n]$. We call such a move on monomials a *transposition step*.

Lemma 18. *Any two monomials in the support of the symmetric determinant* $\det(Z)$ *are linked by a sequence of transposition steps decreasing the distance between the monomials which never leave the support.*

We conjecture that a property inspired by the structure of the determinant and that of stable polynomials holds for all psd-stable polynomials.

Conjecture 19. For any monomial Z^β appearing in a psd-stable polynomial, there is a diagonal monomial Z^α appearing in f which can be reached by a sequence of linear, double and transposition steps which decrease the distance from β to α and which never leave the support of f.

References

[1] G. Blekherman, M. Kummer, R. Sanyal, K. Shu, and S. Sun. *Linear Principal Minor Polynomials: Hyperbolic Determinantal Inequalities and Spectral Containment*. 2021. DOI: 10.48550/ARXIV.2112.13321.

[2] P. Brändén. "Polynomials with the half-plane property and matroid theory". In: *Adv. Math.* 216.1 (2007), pp. 302–320. DOI: 10.1016/j.aim.2007.05.011.

[3] P. Brändén and J. Huh. "Lorentzian polynomials". In: *Ann. of Math. (2)* 192.3 (2020), pp. 821–891. DOI: 10.4007/annals.2020.192.3.4.

[4] Y.-B. Choe, J. G. Oxley, A. D. Sokal, and D. G. Wagner. "Homogeneous multivariate polynomials with the half-plane property". In: vol. 32. 1-2. Special issue on the Tutte polynomial. 2004, pp. 88–187. DOI: 10.1016/S0196-8858(03)00078-2.

[5] P. Dey, S. Gardoll, and T. Theobald. "Conic stability of polynomials and positive maps". In: *J. Pure Appl. Algebra* 225.7 (2021), Paper No. 106610, 19. DOI: 10.1016/j.jpaa.2020.106610.

[6] L. Gårding. "An inequality for hyperbolic polynomials". In: *J. Math. Mech.* 8 (1959), pp. 957–965. DOI: 10.1512/iumj.1959.8.58061.

[7] T. Jörgens and T. Theobald. “Conic stability of polynomials”. In: *Res. Math. Sci.* 5.2 (2018), Paper No. 26, 12. DOI: 10.1007/s40687-018-0144-2.

[8] R. Pemantle. “Hyperbolicity and stable polynomials in combinatorics and probability”. In: *Current developments in mathematics, 2011*. Int. Press, Somerville, MA, 2012, pp. 57–123.

[9] F. Rincón, C. Vinzant, and J. Yu. “Positively hyperbolic varieties, tropicalization, and positroids”. In: *Adv. Math.* 383 (2021), Paper No. 107677, 35. DOI: 10.1016/j.aim.2021.107677.

[10] D. G. Wagner. “Multivariate stable polynomials: theory and applications”. In: *Bull. Amer. Math. Soc. (N.S.)* 48.1 (2011), pp. 53–84. DOI: 10.1090/S0273-0979-2010-01321-5.

Generalised flatness constants: a framework applied in dimension 2

Giulia Codenotti[1*], Thomas Hall[2†], Johannes Hofscheier[3†]

[*]Goethe Universität Frankfurt, Institüt für Mathematik, Germany
[†]School of Mathematical Sciences, University of Nottingham Nottingham, UK

The full version of this work can be found in https://arxiv.org/abs/2110.02770 and will be published elsewhere.

1 Introduction

Let $\mathbb{R}^d$ be the Euclidean space equipped with the Euclidean norm $|\cdot|$. The space of convex bodies, i.e., non-empty compact convex sets, in $\mathbb{R}^d$ is denoted by $\mathcal{K}^d$. Examples of convex bodies are polytopes, convex hulls of finitely many points in $\mathbb{R}^d$. A polytope $P \subset \mathbb{R}^d$ is a lattice polytope if it is the convex hull of finitely many points in the integer lattice $\mathbb{Z}^d$.

We study the lattice width of convex bodies motivated by questions on lattice polytopes and symplectic manifolds. For a convex body $K \subset \mathbb{R}^d$ and a linear form $u \in (\mathbb{Z}^d)^*$, the *width of K along u* is

$$\operatorname{width}_u(K) = \sup_{x,y \in K} |u(x) - u(y)|.$$

The *lattice width of K* (or simply *width of K*), $\operatorname{width}(K)$, is the minimum of its widths $\operatorname{width}_u(K)$ along all $u \in (\mathbb{Z}^d)^* \setminus \{0\}$. Khinchin's celebrated flatness theorem [12] guarantees that for every dimension d there exists a constant which bounds the width of convex bodies whose interiors are disjoint from the integer lattice $\mathbb{Z}^d$. This gives rise to the *classical flatness constant*

$$\operatorname{Flt}_d = \sup\Big\{\operatorname{width}(K) \colon K \in \mathcal{K}^d, \operatorname{int}(K) \cap \mathbb{Z}^d = \emptyset\Big\}.$$

It is conjectured that Flt_d is roughly proportional to d. To the authors' knowledge, the best known upper bound at the time of writing is $\operatorname{Flt}_d \leq O(d^{4/3} \log^a d)$, where a is a constant [4]. Explicit values for Flt_d for low dimensions are scarce: clearly, $\operatorname{Flt}_1 = 1$, and Hurkens has shown that $\operatorname{Flt}_2 = 1 + \frac{2}{\sqrt{3}}$ [10]. However, already Flt_3 is not known: in [6, 2] the bounds $2 + \sqrt{2} \leq \operatorname{Flt}_3 \leq 3.972$ are shown and it is conjectured that $\operatorname{Flt}_3 = 2 + \sqrt{2}$.

In [3], Averkov, Hofscheier, and Nill introduced *generalised flatness constants* that provide a unifying approach to several questions on lattice polytopes and symplectic manifolds. Generalised flatness constants $\operatorname{Flt}_d^A(X)$ depend on the choice of a ring $A \in \{\mathbb{Z}, \mathbb{R}\}$, and the choice of a fixed bounded subset $X \subset \mathbb{R}^d$ and its A-unimodular copies. An *A-unimodular transformation* $T \colon \mathbb{R}^d \to \mathbb{R}^d$ maps an $x \in \mathbb{R}^d$ to $Mx + b$, for some $M \in \mathrm{GL}_d(\mathbb{Z})$ and $b \in A^d$. We say $Y \subset \mathbb{R}^d$ is an *A-unimodular copy of $X \subset \mathbb{R}^d$* if $Y = T(X)$ for some A-unimodular transformation $T \colon \mathbb{R}^d \to \mathbb{R}^d$. Then $\operatorname{Flt}_d^A(X)$ is the supremum over the widths of convex bodies in $\mathbb{R}^d$ whose relative interior does not contain an A-unimodular copy of X, which we call A-X-free convex sets:

$$\operatorname{Flt}_d^A(X) \coloneqq \sup\Big\{\operatorname{width}(K) \colon K \in \mathcal{K}^d, \operatorname{relint}(K) \text{ contains no } A\text{-unimodular copy of } X\Big\}.$$

[1]Email: codenotti@math.uni-frankfurt.de
[2]Email: pmyth1@nottingham.ac.uk
[3]Email: johannes.hofscheier@nottingham.ac.uk. Research of J. H. supported by a Nottingham Research Fellowship from the University of Nottingham

By [3, Theorem 2.1], $\mathrm{Flt}_d^A(X)$ is a well-defined real number. By taking $A = \mathbb{Z}$ and X a lattice point, the usual flatness constant is recovered, i.e., $\mathrm{Flt}_d^{\mathbb{Z}}(\{\mathbf{0}\}) = \mathrm{Flt}_d$, justifying the definition of *generalised* flatness constants. Our first key result in the study of these constants is the following.

Theorem 1. *If $X \subset \mathbb{R}^d$ is a full-dimensional polytope, then every inclusion-maximal A-X-free convex body $K \subset \mathbb{R}^d$ is a polytope.*

Another main result of this work is the computation of generalised flatness constants in dimension 2 when X is the standard simplex $\Delta_2 = \mathrm{conv}(\mathbf{0}, e_1, e_2)$, where e_1, e_2 denotes the standard basis of $\mathbb{Z}^2$.

Theorem 2. *We have $\mathrm{Flt}_2^{\mathbb{R}}(\Delta_2) = 2$ and $\mathrm{Flt}_2^{\mathbb{Z}}(\Delta_2) = \frac{10}{3}$.*

In particular, a convex body whose width is at least $\frac{10}{3}$ is spanning, i.e., the lattice points contained in the convex body affinely generate the ambient lattice. Spanning lattice polytopes turn out to have strong Ehrhart theoretical properties equivalent to ones of IDP polytopes, a much stronger combinatorial assumption on the polytope (see [9]). The search for an effective and sufficient spanning test for lattice polytopes was one of the main motivations for the introduction of generalised $\mathbb{Z}$-flatness constants.

The $\mathbb{R}$-flatness constants are related instead to the computation of the Gromov width of symplectic manifolds. The *Gromov width* of a $2d$-dimensional symplectic manifold (M, ω) (denoted by $c_G(M)$) is the supremum over capacities πr^2 of balls with radius r that can be symplectically embedded in M (see [8]). We are interested in the case when M is a symplectic toric manifold with moment polytope Δ. The explicit computation of Gromov width is still wide open and for symplectic toric manifolds it is not known how to read off the Gromov width from the moment polytope. An important result in [11, Corollary 11.4] can be restated in terms of generalised flatness constants as follows: the Gromov width of a symplectic toric manifold with moment polytope $P \subset \mathbb{R}^d$ is at least $\mathrm{width}(P) \cdot \mathrm{Flt}_d^{\mathbb{R}}(\Delta_d)^{-1}$. Here $\Delta_d \subset \mathbb{R}^d$ denotes the d-dimensional standard simplex. Combining this with Theorem 2 implies a lower bound on the Gromov width of 4-dimensional symplectic toric manifolds.

Theorem 3. *Let (M, ω) be a 4-dimensional symplectic toric manifold with moment polygon Δ. Then the Gromov width $c_G(M)$ of M accepts the following upper and lower bound:*

$$\frac{\mathrm{width}(\Delta)}{2} \leq c_G(M) \leq \mathrm{width}(\Delta).$$

For the lower bound, see also [1], Theorem 5.2. The upper bound is due to [5].

2 A general strategy to compute generalised flatness constants

In this section, we prove foundational observations on generalised flatness constants. We hope the approach introduced here provides an efficient framework for the study of generalised flatness constants in any dimensions and for various choices of X. At the end we outline a general strategy for the computation of generalised flatness constants which we follow to determine $\mathrm{Flt}_2^A(\Delta_2)$ for both $A = \mathbb{Z}, \mathbb{R}$.

The following lemma seems to be decisive for the explicit computation of generalised flatness constants in that it reduces the determination of an upper bound for $\mathrm{Flt}_d^A(X)$ to studying the width of inclusion-maximal A-X-free closed convex sets.

Lemma 4. *Let $X \subset \mathbb{R}^d$ be a non-empty bounded subset. Then every A-X-free convex body is contained in an inclusion-maximal A-X-free closed convex set.*

By Lemma 4, every A-X-free convex body is contained in an inclusion-maximal closed convex A-X-free set. Since the width is monotone with respect to inclusion, we have the following.

$$\sup\left\{\mathrm{width}(K) \;\middle|\; \begin{array}{l} K \text{ inclusion-maximal} \\ A\text{-}X\text{-free convex body} \end{array}\right\} \leq \mathrm{Flt}_d^A(X) \leq \sup\left\{\mathrm{width}(K) \;\middle|\; \begin{array}{l} K \text{ inclusion-maximal} \\ A\text{-}X\text{-free convex set} \end{array}\right\}.$$

That is, an upper bound on the width of inclusion-maximal A-X-free closed convex body $C \subset \mathbb{R}^d$ (including the unbounded ones) gives an upper bound for $\mathrm{Flt}_d^A(X)$, while the width of any inclusion-maximal bounded A-X-free convex set yields a lower bound. A strategy to determine the exact value of generalised flatness constants is to compute these upper and lower bounds and show they agree by studying the explicit values of the width in the two subcases: 1) C is unbounded; 2) C is a convex body.

The following lemma is another important ingredient in the computation of flatness constants.

Lemma 5. *Let $K, X \subset \mathbb{R}^d$ be d-dimensional convex bodies. If K is inclusion-maximal A-X-free, then K contains an A-unimodular copy of X.*

The proof of Theorem 1 for $A = \mathbb{Z}$ is straightforward. We sketch the more involved proof for $A = \mathbb{R}$.

Consider the set of $\mathbb{R}$-unimodular copies S of X which are contained in K. In general, this is an infinite set. We will identify two polytopes in this set if one is the translation of the other. The resulting set of equivalence classes will be denoted by $\mathcal{T}_K(X)$.

Lemma 6. *The set $\mathcal{T}_K(X)$ of $\mathbb{R}$-translation equivalence classes of $\mathbb{R}$-unimodular copies of X contained in K is finite.*

For subsets $A, B \subset \mathbb{R}^d$ the *Minkowski difference* $A \div B$ is the set of translation vectors that move B into A:

$$A \div B = \{x \in \mathbb{R}^d \colon B + x \subset A\} = \bigcap_{b \in B} (A - b).$$

Let $S \subset \mathbb{R}^d$ be a fixed $\mathbb{Z}$-unimodular copy of X. Studying the Minkowski difference $K \div S$ is key to proving Theorem 1, since they allow to characterise when exactly K is $\mathbb{R}$-X-free:

Lemma 7. *An $\mathbb{R}$-translation of a $\mathbb{Z}$-unimodular copy S of X is contained in the interior of K if and only if $\dim(K \div S) = d$. Thus K is $\mathbb{R}$-X-free if and only if $\dim(K \div S) < d$ for all $\mathbb{Z}$-unimodular copies S of X.*

Fix an inclusion-maximal $\mathbb{R}$-X-free convex body $K \subset \mathbb{R}^d$. We want to show that K is a polytope. As a first step, we approximated K by a polytope $P \subset \mathbb{R}^d$ containing K such that up to real translations K and P contain the same $\mathbb{Z}$-unimodular copies of X.

Lemma 8. *For a d-dimensional convex body $K \subset \mathbb{R}^d$, there exists a d-dimensional convex polytope $P \subset \mathbb{R}^d$ which contains K such that $\mathcal{T}_K(X) = \mathcal{T}_P(X)$.*

The polytope P which contains K in Theorem 8 isn't necessarily $\mathbb{R}$-X-free. To conclude the proof of Theorem 1 for $A = \mathbb{Z}$ we show that it is always possible to intersect P with further half-spaces to obtain a new polytope which is $\mathbb{R}$-X-free.

3 The $\mathbb{Z}$-flatness constant of Δ_2

The following definition is the key to characterising inclusion-maximal $\mathbb{Z}$-Δ_d-free polytopes.

Definition 9. A facet F of a full-dimensional polytope $P \subset \mathbb{R}^d$ is said to be $\mathbb{Z}$-Δ_d-*locked* if there exists a $\mathbb{Z}$-unimodular copy T of Δ_d contained in P such that $T \cap \mathrm{relint}(F) \neq \emptyset$ and $V(T) \setminus \mathrm{relint}(F) \subset \mathrm{int}(P)$, where $V(T)$ denotes the set of vertices of T. Notice that T gives rise to lattice points in the relative interior of F, namely $V(T) \cap \mathrm{relint}(F)$. Such lattice points are called *locking points*.

The definition of locked facet is crafted so that if a point is added beyond any locked facet, the resulting polytope is no longer a $\mathbb{Z}$-Δ_d-free polytope. The following shows that this is a characterization.

Proposition 10. *Let $P \subset \mathbb{R}^d$ be a $\mathbb{Z}$-Δ_d-free polytope. Then P is inclusion-maximal if and only if all its facets are locked.*

We now focus on dimension $d = 2$, where our goal is to show the following theorem.

Theorem 11 (Case $A = \mathbb{Z}$ of Theorem 2)**.** $\mathrm{Flt}_2^{\mathbb{Z}}(\Delta_2) = \frac{10}{3}$.

By Theorem 4, to prove the theorem it is enough to show that any inclusion-maximal $\mathbb{Z}$-Δ_2-free convex set has width at most $\frac{10}{3}$ and to provide an example of a $\mathbb{Z}$-Δ_2-free polygon of that width. Unbounded full-dimensional inclusion-maximal $\mathbb{Z}$-Δ_2-free convex sets have width 2. Thus we are left with determining the maximum width of inclusion-maximal $\mathbb{Z}$-Δ_2-free convex bodies, which are polygons by Theorem 1. The rest of the section is devoted to proving that this width is $\frac{10}{3}$.

Theorem 10 guarantees that any inclusion-maximal $\mathbb{Z}$-Δ_2-free polygon contains at least one interior lattice point, since otherwise it is impossible for its facets to be locked. We first deal with the case of exactly one interior lattice point.

Proposition 12. *If $P \subset \mathbb{R}^2$ is a maximal $\mathbb{Z}$-Δ_2-free polygon with $|\mathrm{int}(P) \cap \mathbb{Z}^2| = 1$, then* $\mathrm{width}(P) \leq 3$.

Now suppose the polygon P contains at least two interior lattice points, which must then necessarily be collinear.

Theorem 13. *If $P \subset \mathbb{R}^2$ is a $\mathbb{Z}$-Δ_2-free polygon with $|\mathrm{int}(P) \cap \mathbb{Z}^2| \geq 2$, then* $\mathrm{width}(P) \leq \frac{10}{3}$. *Equality is only achieved (up to $\mathbb{Z}$-unimodular transformations) by* $\mathrm{conv}\Big(\frac{1}{3}e_1 + \frac{5}{3}e_2, -\frac{4}{3}e_1 - \frac{5}{3}e_2, 2e_1\Big)$, *which contains exactly* 2 *interior lattice points.*

In order to prove Theorem 13, it suffices to study inclusion-maximal $\mathbb{Z}$-Δ_2-free polygons which have at least two interior lattice points. We can show that the inclusion-maximal polygon P is either a triangle or a quadrilateral. We further subdivide the case where P is a quadrilateral into two subcases: one is when the convex hull P' of the four locking points in the interior of each facet of P is unimodularly equivalent to a rectangle, the other when P' is equivalent to a cross-polygon.

We then show by theoretical arguments that the width of all quadrilaterals circumscribed around a rectangle is at most 3 and that the width of those circumscribed around a cross-polygon is strictly less than $\frac{10}{3}$. To treat the case of triangles, we find a finite list of possible triples (A, B, C) of locking points, and to bound the width of polygons whose facets are locked by each fixed triple (A, B, C), we employ a computer assisted strategy together with an approach that Hurkens has used to compute the classical flatness constant in 2 dimensions [10]. Let X, Y and Z be the vertices of the triangle P. For A, B and C fixed, there are formulas for X, Y and Z in terms of real parameters λ, μ, and ν. In fact, the coordinates of the pairwise differences of the vertices X, Y and Z are rational functions with a *linear* numerator and denominator equal to $\lambda\mu\nu + \bar{\lambda}\bar{\mu}\bar{\nu}$:

$$\begin{bmatrix} X - Y \\ Y - Z \\ Z - X \end{bmatrix} = \frac{1}{\lambda\mu\nu + \bar{\lambda}\bar{\mu}\bar{\nu}} \begin{bmatrix} -\mu & \bar{\lambda} & -1 + \lambda + \mu \\ -1 + \mu + \nu & -\nu & \bar{\mu} \\ \bar{\nu} & -1 + \lambda + \nu & -\lambda \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}. \tag{1}$$

The slopes of the facets of P are thus rational functions with linear numerator and linear denominator in terms of the parameters λ, μ, and ν, and are constrained by conditions on the lattice points. We obtain a polytope $Q \subset [0, 1]^3$ of admissible triples (λ, μ, ν). Next, we express the width of P in a chosen direction in terms of the parameters λ, μ, and ν. Wherever the width in a fixed direction is achieved by the same two vertices, it is a linear function (in λ, μ, and ν) divided by $\delta := \lambda\mu\nu + \bar{\lambda}\bar{\mu}\bar{\nu}$. Our strategy includes choosing directions "ad hoc" such that:

1) The width in one such direction is achieved on the same pair of vertices for all parameters $\lambda, \mu, \nu \in Q$, so that the width of P accepts an upper bound of the form

$$\frac{\min\{\ell_1(\lambda, \mu, \nu), \ldots, \ell_r(\lambda, \mu, \nu)\}}{\delta} \qquad \text{where the } \ell_i\text{'s are linear forms in terms of } \lambda, \mu, \text{ and } \nu.$$

2) The maximum over Q of this function is at most $\frac{10}{3}$.

The computations are carried out with polymake [7] and Mathematica [13]. The code can be found at https://github.com/jhofscheier/gen-flat-const-dim2

Where the upper bound of $\frac{10}{3}$ is achieved, we determine the respective extremal points $(\lambda, \mu, \nu) \in Q$ and show that these correspond to triangles unimodularly equivalent to $\operatorname{conv}\left(\frac{1}{3}e_1 + \frac{5}{3}e_2, -\frac{4}{3}e_1 - \frac{5}{3}e_2, 2e_1\right)$.

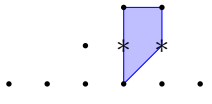
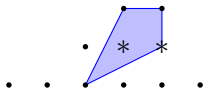
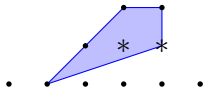
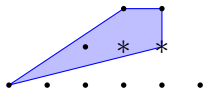
$\operatorname{conv}(\mathbf{0}, e_1, A, B, C)$	width	width directions	vertices of maximiser
	$\leq \frac{10}{3}$	$e_1^*, e_2^*, e_1^* - e_2^*$	$\frac{1}{3}\begin{bmatrix} -4 & 1 & 6 \\ -5 & 5 & 0 \end{bmatrix}$
	$\leq \frac{2}{\sqrt{7}-2}$	$e_1^*, e_2^*, e_1^* - e_2^*$	$\frac{1}{3\sqrt{7}}\begin{bmatrix} -6-3\sqrt{7} & 8+\sqrt{7} & 1+2\sqrt{7} \\ -9 & -2-\sqrt{7} & 5+4\sqrt{7} \end{bmatrix}$
	$< \frac{10}{3}$	$e_2^*, e_1^* - e_2^*$	$\frac{1}{3}\begin{bmatrix} -12 & 3 & 3 \\ -5 & 5 & 0 \end{bmatrix}$
	< 3	e_2^*	$\frac{1}{2}\begin{bmatrix} -10 & 2 & 2 \\ -3 & 3 & 0 \end{bmatrix}$

Table 1: Up to symmetry, the possible triples of locking points (A, B, C), shown as the black-dot vertices of the blue quadrilaterals in the left column. The maximum width and the directions where width is achieved for a triangle circumscribed to (A, B, C) are listed in the next two columns, and the rightmost column are matrices whose columns are the vertices of the maximising triangle.

4 $\mathbb{R}$-Generalised Flatness Constant of Δ_2

The first goal this section is to prove the following theorem (case $A = \mathbb{R}$ of Theorem 2).

Theorem 14 (Case $A = \mathbb{R}$ of Theorem 2)**.** $\operatorname{Flt}_2^{\mathbb{R}}(\Delta_2) = 2$.

The cross-polygon $\Diamond_2 := \operatorname{conv}(\pm e_1, \pm e_2)$ is $\mathbb{R}$-Δ_2-free and has width 2, and hence $\operatorname{Flt}_2^{\mathbb{R}}(\Delta_2) \geq 2$. To prove Theorem 14, we need to bound the $\mathbb{R}$-flatness constant of Δ_2 from above by 2. By Lemma 4, it suffices to bound the width of inclusion-maximal $\mathbb{R}$-Δ_2-free closed convex sets C by 2. If C is unbounded, its width is bounded by 1. Hence, it remains to study the width of the bounded C's which, by Theorem 1, are polygons. Our strategy is to show that any polygon $P \subset \mathbb{R}^2$ with width greater than 2 is not $\mathbb{R}$-Δ_2-free. A key ingredient in the proof is the notion of rational diameter (we use the following shorthand notation for horizontal line segments $S \subset \mathbb{R}^d$: $[x, y] := \operatorname{conv}(xe_1, ye_1)$):

Definition 15. Let $K \subset \mathbb{R}^d$ be a convex body. The *rational diameter of* K is the largest dilation of an $\mathbb{R}$-unimodular copy of the unit segment $[0, 1]$ which is contained in K, i.e.

$$l(K) := \max\{l \in \mathbb{R}_{\geq 0} \colon lS \subseteq K \text{ for some } \mathbb{R}\text{-unimodular copy } S \text{ of } [0, 1]\}.$$

To prove Theorem 14 we are only interested in polygons of width strictly larger than 2. The following lemma shows that we then only need to consider polygons with rational diameter $l > 1$.

Lemma 16. *Let $P \subset \mathbb{R}^2$ be a polygon with rational diameter $l = 1 + 2a > 0$ (that is, $a > -\frac{1}{2}$), achieved with $S = [-a, 1 + a] \subseteq P$. Then $P \subset \{(x, y) \in \mathbb{R}^2 \colon -l \leq y \leq l\}$.*

The following lemma will allow us to also bound the rational diameter from above.

Lemma 17. *Let $P \subset \mathbb{R}^2$ be an $\mathbb{R}$-Δ_2-free polygon with rational diameter $l = 1 + 2a > 1$ (that is, $a > 0$), achieved with $S = [-a, 1 + a] \subseteq P$. Then $\operatorname{int}(P)$ is disjoint from the segments $[-a, a] + (b, \pm 1)$, for all $b \in \mathbb{Z}$. In particular, if $a \geq \frac{1}{2}$ then $P \subset \{(x, y) \in \mathbb{R}^2 \colon -1 \leq y \leq 1\}$ and $\operatorname{width}(P) \leq 2$.*

To prove Theorem 14, we use the bounds on the lattice diameter from the previous lemmas and the lower bound on the width to show that certain areas of the plane are disjoint from the polygon P. Eventually this allows us to bound the polygon so tightly that we reach a contradiction.

We note that in the case of $\mathbb{R}$-flatness we do not obtain a complete characterization of width-maximising $\mathbb{R}$-Δ_2-free polygons. We only know two examples: the cross-polygon $\operatorname{conv}(\pm e_1, \pm e_2)$ and the triangle $\operatorname{conv}(e_1, e_2, -e_1 - e_2)$. Indeed, the study of inclusion-maximal $\mathbb{R}$-Δ_2-free is more intricate than in the case of $\mathbb{Z}$-flatness. We give a notion of $\mathbb{R}$-locked facets which allows us to construct a family of maximal $\mathbb{R}$-Δ_2-free quadrilaterals: all parallelograms circumscribed around a unit square $[0, 1]^2$. However, there are many examples of maximal $\mathbb{R}$-Δ_2-free quadrilaterals which we do not know how to characterise, and examples of maximal $\mathbb{R}$-Δ_2-free polygons with more facets.

References

[1] F. Ambro and A. Ito. "Successive minima of line bundles". In: *Adv. Math.* 365 (2020), pp. 107045, 38. DOI: 10.1016/j.aim.2020.107045.

[2] G. Averkov, G. Codenotti, A. Macchia, and F. Santos. "A local maximizer for lattice width of 3-dimensional hollow bodies". In: *Discrete Appl. Math.* 298 (2021), pp. 129–142. DOI: 10.1016/j.dam.2021.04.009.

[3] G. Averkov, J. Hofscheier, and B. Nill. "Generalized flatness constants, spanning lattice polytopes, and the Gromov width". In: *manuscripta mathematica* (2021). DOI: 10.1007/s00229-021-01363-x.

[4] W. Banaszczyk, A. E. Litvak, A. Pajor, and S. J. Szarek. "The flatness theorem for nonsymmetric convex bodies via the local theory of Banach spaces". In: *Math. Oper. Res.* 24.3 (1999), pp. 728–750. DOI: 10.1287/moor.24.3.728.

[5] J. Chaidez and B. Wormleighton. *ECH embedding obstructions for rational surfaces*. Preprint. 2020. DOI: 10.48550/ARXIV.2008.10125.

[6] G. Codenotti and F. Santos. "Hollow polytopes of large width". In: *Proc. Amer. Math. Soc.* 148.2 (2020), pp. 835–850. DOI: 10.1090/proc/14721.

[7] E. Gawrilow and M. Joswig. "polymake: a framework for analyzing convex polytopes". In: *Polytopes—combinatorics and computation (Oberwolfach, 1997)*. Vol. 29. DMV Sem. Birkhäuser, Basel, 2000, pp. 43–73.

[8] M. Gromov. "Pseudo holomorphic curves in symplectic manifolds". In: *Invent. Math.* 82.2 (1985), pp. 307–347. DOI: 10.1007/BF01388806.

[9] J. Hofscheier, L. Katthän, and B. Nill. "Ehrhart theory of spanning lattice polytopes". In: *Int. Math. Res. Not. IMRN* 19 (2018), pp. 5947–5973. DOI: 10.1093/imrn/rnx065.

[10] C. A. J. Hurkens. "Blowing up convex sets in the plane". In: *Linear Algebra Appl.* 134 (1990), pp. 121–128. DOI: 10.1016/0024-3795(90)90010-A.

[11] K. Kaveh. "Toric degenerations and symplectic geometry of smooth projective varieties". In: *J. Lond. Math. Soc. (2)* 99.2 (2019), pp. 377–402. DOI: 10.1112/jlms.12173.

[12] A. Y. Khinchine. “A quantitative formulation of the approximation theory of Kronecker”. In: *Izvestiya Akad. Nauk SSSR. Ser. Mat.* 12 (1948), pp. 113–122.

[13] Wolfram Research, Inc. *Mathematica*. Version 12.3. `https://www.wolfram.com`. 2021.

Multitriangulations and tropical Pfaffians

Luis Crespo Ruiz[1], Francisco Santos[2]

Departamento de Matemáticas, Estadística y Computación, Universidad de Cantabria, 39005 Santander, Spain

The full version of this work can be found in [2] and will be published elsewhere. This research is supported by grant PID2019-106188GB-I00 funded by MCIN/AEI/10.13039/501100011033, by FPU19/04163 of the Spanish Government, and by project CLaPPo (21.SI03.64658) of Universidad de Cantabria and Banco Santander.

Abstract

The k-asociahedron is a simplicial complex whose facets correspond to k-triangulations of the n-gon, known to be homeomorphic to a sphere of dimension $k(n-2k-1)-1$ and conjectured to be polytopal by Jonsson, among others. (The case or $k=1$ is the classical associahedron of dimension $n-4$). We show that it can be obtained by intersecting the tropical variety of Pfaffians with the orthant of "four-point positive" weights. We hope this to be a step towards realizing it as a polytope.

1 k-triangulations

Definition 1 (*See e.g. [3]*)**.** Let $n > 2k$ be two positive integers. A subset $T \subseteq \binom{[n]}{2}$ of diagonals of the convex n-gon is called $(k+1)$-free if no $(k+1)$ diagonals in T pairwise cross. The maximal $(k+1)$-free graphs are called *k-triangulations* or multi-triangulations.

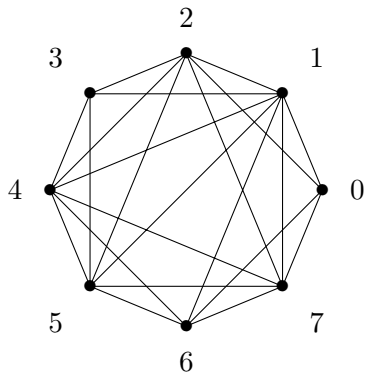


Figure 1: A 2-triangulation of the 8-gon. As expected, it has 22 edges

We are interested in the abstract simplicial complex $\mathcal{A}ss_k(n)$ on the vertex set $\binom{[n]}{2}$ whose faces are $(k+1)$-free subsets and whose facets are k-triangulations. All *k-triangulations* are known to have cardinality $k(2n-2k-1)$ [3, 5]. That is, $\mathcal{A}ss_k(n)$ is a pure simplicial complex.

If an edge $\{i,j\}$ satisfies $|i-j| \leq k$ (where indices are taken modulo n, and distance is measured cyclically), then it lies in every k-triangulation. We call these edges *irrelevant* and call the face of $\mathcal{A}ss_k(n)$ they span the *irrelevant face*. We can thus define the reduced complex, $\overline{\mathcal{A}ss}_k(n)$, the faces

[1]Email: luis.cresporuiz@unican.es
[2]Email: francisco.santos@unican.es

of which are the $(k+1)$-free sets of relevant edges. The exact relation between $\mathcal{A}ss_k(n)$ and $\overline{\mathcal{A}ss}_k(n)$ is that the former is the join of the latter with the irrelevant face, and hence the latter is the link of the former at the irrelevant face. Based on the fact that $\overline{\mathcal{A}ss}_1(n)$ is the face poset of the polar of the standard associahedron, we define:

Definition 2. We call $\overline{\mathcal{A}ss}_k(n)$ the *k-associahedron* of parameters n, k. We refer to $\mathcal{A}ss_k(n)$ as the *extended k-associahedron*.

Jonsson [5] proved that $\overline{\mathcal{A}ss}_k(n)$ is a shellable simplicial sphere of dimension $k(n-2k-1)-1$, and conjectured it to be polytopal. This conjecture is one of the motivations for this work.

2 Tropical varieties

Let us briefly recall what the tropical variety of an ideal is. Let $f \in \mathbb{K}[x_1, \ldots, x_N]$ be a polynomial. Each vector $d \in \mathbb{R}^N$, considered as giving weights to the variables, defines an *initial form* $\mathrm{in}_d(f)$. This initial form is obtained by sorting the terms of f by the sum of the weights of their variables (weighted degree), and neglecting those of non-maximum sum. If d is "generic" then $\mathrm{in}_d(f)$ is a single monomial, but we are interested in the opposite case. The *tropical hypersurface of f*, denoted $\mathrm{trop}(f)$, is the set of vectors d for which at least two monomials attain the maximum weight. Put differently, $\mathrm{trop}(f) \subset \mathbb{R}^N$ equals the codimension-one skeleton of the normal fan of the Newton polytope of f.

Definition 3. A *tropical prevariety* is any finite intersection of tropical hypersurfaces. The *tropical variety* of an ideal $I \subset \mathbb{K}[x_1, \ldots, x_N]$ is $\mathrm{trop}(I) := \cap_{f \in I} \mathrm{trop}(f)$, that is, the set of $d \in \mathbb{R}^N$ for which $\mathrm{in}_d(I)$ does not contain any monomial. If $\mathbb{K} = \mathbb{C}$ (or any field containing $\mathbb{R}$), the *totally positive part* of $\mathrm{trop}(I)$, denoted $\mathrm{trop}^+(I)$, equals the set of $d \in \mathbb{R}^N$ for which $\mathrm{in}_d(I)$ does not contain any polynomial with all coefficients real and positive.

Although a tropical variety is defined as an infinite intersection of tropical hypersurfaces, for each I a certain finite subset is enough; that is, every tropical variety is a prevariety [8, Thm. 2.6.5]. However, not every generating set of I is enough, not even a universal Gröbner basis.

3 The ideal of Pfaffians of degree $k+1$

The determinant of an antisymmetric matrix of even size $2k$ with indeterminate entries is the perfect square of a homogeneous polynomial of degree k, called the *Pfaffian*. It has $2k!!$ terms, corresponding to the (perfect) matchings among the $2k$ labels for rows and columns, all with coefficient ± 1 depending on the *parity* of each matching [1, 7, 11]. For example, for $k = 2$ we get

$$\begin{vmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{vmatrix} = (x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23})^2. \tag{1}$$

The 3 terms inside the square correspond to the 3 matchings among 4 points, shown in Fig. 2.

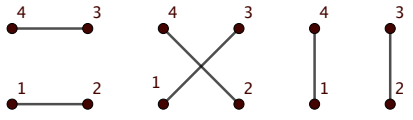


Figure 2: The three matchings among four points, corresponding to the Pfaffian in Eq. (1)

For each $n \geq 2k+2$, let $I_k(n)$ be the ideal in $\mathbb{K}[x_{i,j}, \{i,j\} \in \binom{[n]}{2}]$ generated by all the *principal Pfaffians of degree* $k+1$ of an antisymmetric matrix M of size n. That is, for each subset $U \in \binom{[n]}{2k+2}$, we consider the Pfaffian of the principal minor of M labeled by U. Let $\mathcal{P}f_k(n) \subset \mathbb{K}^{\binom{n}{2}}$ be the corresponding variety. That is, points in $\mathcal{P}f_k(n)$ correspond to antisymmetric $n \times n$ matrices with coefficients in $\mathbb{K}$ and of rank at most $2k$. Since Pfaffians of degree two (see Eq. (1)) coincide with the quadratic Plücker relations, $I_1(n)$ equals the Plücker ideal defining the Grassmannian $\mathcal{G}r_2(n)$ in $\mathbb{K}^{\binom{[n]}{2}}$. See, e.g., [10, Remark 3.22]. In this $k=1$ case, Speyer and Williams [12] have shown that the associahedron arises as a subfan of the tropical variety of $I_1(n)$, as we now recall.

Definition 4. We say that a weight vector $v \in \mathbb{R}^{\binom{[n]}{2}}$ is *four-point positive* (abbreviated *fp-positive*) if for all $1 \leq a < a' < b < b' \leq n$ we have that

$$v_{a,b} + v_{a',b'} \geq \max\{v_{a,a'} + v_{b,b'}, v_{a,b'} + v_{a',b}\}. \tag{2}$$

We denote by FP_n the subset of $\mathbb{R}^{\binom{[n]}{2}}$ consisting of fp-positive vectors.

Observe that FP_n equals the space of weights that select as initial term, in each Pfaffian of four points, the matching that has a crossing. It can also be interpreted as the weights that represent *separation vectors* among sides of the n-gon, or as weights that are monotone with respect to crossing-increasing swaps among perfect matchings of each $U \in \binom{[n]}{2k+2}$. Geometrically, FP_n is a cone linearly isomorphic to an orthant plus a linear space. See details in [2, Sect. 3.1].

Theorem 5 ([12, Section 5])**.** *The intersection* $\mathrm{trop}(I_1(n)) \cap \mathrm{FP}_n$ *is a simplicial fan isomorphic to (the cone over) the extended associahedron* $\mathcal{A}ss_1(n)$*.*

4 Our results for arbitrary k

Since these relations are a universal Gröbner basis for the ideal $I(\mathcal{G}r_2(n)) = I(\mathcal{P}f_1(n)) = I_1(n)$, the initial ideal of $\mathcal{P}f_1(n)$ with respect to any generic fp-positive weight is generated by the crossing monomials of degree two.

Our main result generalizes Theorem 5 to arbitrary k. The starting point is to show that Pfaffians form a Gröbner basis of the ideal they generate, for any weight vector in FP_n. This generalizes the main result of [6], who prove it for a particular lexicographic weight vector:

Theorem 6. *With respect to any weight vector* $v \in \mathrm{FP}_n$*, Pfaffians are a Gröbner basis for the ideal* $I_k(n)$*. Moreover, if* v *lies in the interior of* FP_n *then* $\mathrm{in}_v(I_k(n))$ *is the monomial ideal generated by* $(k+1)$*-crossings. That is, it is the Stanley-Reisner ideal [9] of* $\mathcal{A}ss_k(n)$*.*

Let us denote as $\mathrm{Grob}_k(n)$ the Gröbner cone of $I_k(n)$ that selects the $(k+1)$-crossing term in each Pfaffian. The Theorem says that $\mathrm{FP}_n \subseteq \mathrm{Grob}_k(n)$. For $k=1$ we actually have an equality, $\mathrm{FP}_n = \mathrm{Grob}_1(n)$, but for $k \geq 2$ this is no longer true. Yet, the Gröbner cone is still linearly isomorphic to an orthant (except for $n = 2k+2$, where there is a single Pfaffian and the Gröbner cone equals a normal cone in its Newton polytope). Explicit descriptions of $\mathrm{Grob}_k(n)$ by rays and by inequalities appear in the full version of this paper [2].

We now denote as $\mathrm{Pf}_k(n) \subset \mathbb{R}^{\binom{[n]}{2}}$ the intersection of the tropical hypersurfaces of Pfaffians of degree k. This is by definition a tropical *prevariety* containing $\mathrm{trop}(I_k(n))$, but it may not coincide with it. Put differently, we do not know whether Pfaffians are a *tropical basis* for the ideal $I_k(n)$.

In the light of Theorem 6, it makes sense to look at the part of $\mathrm{Pf}_k(n)$ defined by fp-positive vectors. That is, we define $\mathrm{Pf}_k^+(n) := \mathrm{Pf}_k(n) \cap \mathrm{FP}_n$.

We prove that for any $v \in \mathrm{FP}_n$, being in $\mathrm{Pf}_k^+(n)$ is equivalent to the positivity equations (2) being satisfied with equality except in a $(k+1)$-free set [2, Theorem 4.3]. Moreover, when this happens, v can be proved to be in $\mathrm{trop}(I_k(n))$ [2, Corollary 4.5]. Thus:

Theorem 7. *1.* FP_n *is linearly isomorphic to an orthant.*

2. $\mathrm{Pf}_k^+(n) = \mathrm{trop}(I_k(n)) \cap \mathrm{FP}_n \subset \mathrm{trop}^+(I_k(n))$.

3. $\mathrm{Pf}_k^+(n)$ *is the union of the faces of the orthant* FP_n *corresponding to* $(k+1)$*-free graphs.*

That is, $\mathrm{Pf}_k^+(n)$ is embedded in FP_n as (the cone over) the k-associahedron $\mathcal{A}ss_k(n)$.

Once we have this result, in order to realize the k-associahedron as a complete (and hopefully polytopal) fan, it would suffice to find a projection $\mathbb{R}^{\binom{[n]}{2}} \to \mathbb{R}^{k(2n-2k-1)}$ that is injective in $\mathrm{Pf}_k^+(n)$. This idea works nicely for the case $k = 1$, as we show in the next section, but we have not managed to implement it for higher k.

5 Recovering the g-vector fan of the classical associahedron ($k = 1$)

In this section, let $k = 1$ and let T be a particular triangulation of the n-gon, which we call the seed triangulation. The special property of the case $k = 1$ that we are going to use is that the projection that remembers only the coordinates corresponding to edges used in T is injective on $\mathrm{Pf}_1^+(n)$. This is not true for $k = 2$ or higher.

Lemma 8. *For every* $(v_{i,j})_{i,j} \in \mathrm{Pf}_1^+(n)$*, knowing the entries of* v *corresponding to* T*, we can recover all other entries. That is, the projection* $\pi : \mathrm{Pf}_1^+(n) \to \mathbb{R}^T \cong \mathbb{R}^{2n-3}$ *that restricts each vector* $(v_{i,j})_{i,j}$ *to the entries with* $\{i,j\} \in T$ *is injective.*

That is, π embeds $\mathrm{Pf}_1^+(n)$ as a full-dimensional fan $\pi(\mathrm{Pf}_1^+(n)) \subset \mathbb{R}^T \cong \mathbb{R}^{2n-3}$. We are interested in a second projection

$$\phi : \mathbb{R}^T \to \mathbb{R}^{\overline{T}} \cong \mathbb{R}^{n-3}$$

that sends the irrelevant face of $\pi(\mathrm{Pf}_1^+(n))$ to zero, so that $\phi(\pi(\mathrm{Pf}_1^+(n)))$ is a fan isomorphic to the link of the irrelevant face in $\pi(\mathrm{Pf}_1^+(n))$, that is, isomorphic to $\overline{\mathcal{A}ss}_1(n)$, the normal fan of the associahedron. Here, $\overline{T}$ denotes the relevant part (the $n-3$ diagonals) of T.

Corollary 9. *The projection*

$$\phi \circ \pi : \mathrm{Pf}_1^+(n) \to \mathbb{R}^{\overline{T}} \cong \mathbb{R}^{n-3}$$

gives a realization of the associahedron $\overline{\mathcal{A}ss}_1(n)$ *as a complete fan.*

We can give an explicit description of this associahedron using the so called **g**-vectors. For any given diagonal δ we define the following *crossing sign* of $\{a,b\}$ with respect to δ and the **g**-vector of $\{a,b\}$ with respect to T as follows:

Definition 10 (See [4, Proposition 33])**.** Let δ be a diagonal in $\overline{T}$ and $\{a,b\} \in \binom{[n]}{2}$. Let $q(\delta)$ be the quadrilateral in T consisting of δ and its two adjacent triangles. We define the *crossing sign of* $\{i,j\}$ *with respect to* δ *in* T

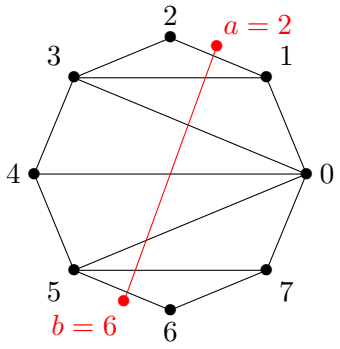
$$\varepsilon(\delta \in T, \{a,b\}) := \begin{cases} +1 & \text{if } \{a,b\} \text{ crosses two opposite sides and the diagonal of } q(\delta) \text{ forming a } \mathsf{Z} \\ -1 & \text{if } \{a,b\} \text{ crosses two opposite sides and the diagonal of } q(\delta) \text{ forming a } \Sigma \\ 0 & \text{otherwise} \end{cases}$$

We define the **g**-*vector of* $\{a,b\}$ *with respect to* T as

$$\mathbf{g}(T, \{a,b\}) := (\varepsilon(\delta \in T, \{a,b\}))_{\delta \in \overline{T}}$$

For example, for the following triangulation and the edge $\{2,6\}$, we have

$$\begin{aligned}\varepsilon(\{1,3\} \in T, \{2,6\}) = \varepsilon(\{0,5\} \in T, \{2,6\}) &= 1\\ \varepsilon(\{0,3\} \in T, \{2,6\}) &= -1\\ \varepsilon(\{0,4\} \in T, \{2,6\}) = \varepsilon(\{5,7\} \in T, \{2,6\}) &= 0\end{aligned}$$



Hohlweg, Pilaud and Stella [4] consider the **g**-vector fan obtained considering as cones all the possible clusters (which, in type A are the triangulations) and taking as generators the **g**-vectors for a fixed but arbitrary seed triangulation T. The main result of [4] is that these fans are polytopal. It turns out that these fans are linearly isomorphic to the ones of Corollary 9:

Theorem 11. *Let $\Sigma_T = im(\phi \circ \pi)$ be the associahedral fan in $\mathbb{R}^{n-3}$ of Corollary 9 for a certain seed triangulation T.*

In the basis of $\mathbb{R}^{n-3}$ consisting of the rays corresponding to the diagonals of $\overline{T}$ we have that Σ_T equals the **g***-vector fan of T.*

Proposition 12 ([4])**.** *For every T, the fan Σ_T is polytopal.*

In conclusion, we have that projecting $\mathrm{Pf}_1^+(n)$ to the $n-3$ coordinates of the diagonals in T gives a realization of the $(n-3)$-associahedron in $\mathbb{R}^{n-3}$ with the following properties:

1. It has $n-3$ pairs of parallel facets, each pair consisting of the facet corresponding to a diagonal $\{a,b\}$ of T and its rotation $\{a+1,b+1\}$.
2. Taking the normals to those $n-3$ pairs as a linear basis for $\mathbb{R}^{n-3}$, all other facet normals lie in $\{0,\pm 1\}^{n-3}$.

References

[1] A. Cayley. *On the theory of permutants.* Vol. VII. Reprinted in Collected mathematical papers. Volume 2. 1852, pp. 40–51.

[2] L. Crespo Ruiz and F. Santos. *Multitriangulations and tropical Pfaffians.* 2022. DOI: 10.48550/ARXIV.2203.04633.

[3] A. Dress, J. H. Koolen, and V. Moulton. "On line arrangements in the hyperbolic plane". In: *European J. Combin.* 23.5 (2002), pp. 549–557. DOI: 10.1006/eujc.2002.0582.

[4] C. Hohlweg, V. Pilaud, and S. Stella. "Polytopal realizations of finite type **g**-vector fans". In: *Adv. Math.* 328 (2018), pp. 713–749. DOI: 10.1016/j.aim.2018.01.019.

[5] J. Jonsson. *Generalized triangulations of the n-gon.* Unpublished manuscript. An abstract was included in: Topological and Geometric Combinatorics, 6–12 April 2003, Mathematisches Forschungsinstitut Oberwolfach, Report No. 16/2003, http://www.mfo.de/programme/schedule/2003/15/Report16_2003.pdf. 2003.

[6] J. Jonsson and V. Welker. "A spherical initial ideal for Pfaffians". In: *Illinois J. Math.* 51.4 (2007), pp. 1397–1407.

[7] L. Lovász and M. D. Plummer. *Matching theory*. Vol. 121. Annals of Discrete Mathematics, 29. North-Holland Publishing Co., Amsterdam, 1986, pp. xxvii+544.

[8] D. Maclagan and B. Sturmfels. *Introduction to tropical geometry*. Vol. 161. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2015, pp. xii+363. DOI: 10.1090/gsm/161.

[9] E. Miller and B. Sturmfels. *Combinatorial commutative algebra*. Vol. 227. Graduate Texts in Mathematics. Springer-Verlag, New York, 2005, pp. xiv+417.

[10] L. Pachter and B. Sturmfels. *Algebraic statistics for computational biology*. Cambridge Univ. Press, New York, 2005. DOI: 10.1017/CBO9780511610684.

[11] G. Rote. "Division-free algorithms for the determinant and the Pfaffian: algebraic and combinatorial approaches". In: *Computational discrete mathematics*. Vol. 2122. Lecture Notes in Comput. Sci. Springer, Berlin, 2001, pp. 119–135. DOI: 10.1007/3-540-45506-X_9.

[12] D. Speyer and L. Williams. "The tropical totally positive Grassmannian". In: *J. Algebraic Combin.* 22.2 (2005), pp. 189–210. DOI: 10.1007/s10801-005-2513-3.

A parallel between the descriptive complexities of finite groups and Latin square graphs

Anuj Dawar[1*], Danny Vagnozzi[2†]

[*]University of Cambridge
[†]University of Edinburgh

Abstract

We establish a tight connection between the Weisfeiler-Leman dimension of finite groups and that of Latin square graphs of Cayley tables. Hence, we discuss the consequences of a conjecture by Emms et al. regarding a polynomial time graph invariant arising from quantum walks.

1 Introduction

The group isomorphism problem consists in deciding whether there is a product preserving bijection between two groups given as multiplication tables. Despite the concept of isomorphism between groups being fundamental in most areas of mathematics, our understanding of the computational complexity of deciding group isomorphism is still riddled with gaps. Like other problems such as graph isomorphism, there are no known polynomial time algorithms nor the structural evidence to suggest whether such an algorithm even exists. However, it is important to note that while the difficulty of deciding isomorphism of graphs increases with size, this is not necessarily the case for groups. Indeed, the number of isomorphism types of groups of a given order depends heavily on the number of prime factors of the order as well as their multiplicities. This peculiarity seems to give group isomorphism a nature of its own.

Given two groups of order n, the most efficient known algorithms can decide the isomorphism thereof in time $n^{O(\log(n))}$. In fact, this is the worst case running time for early approaches such as Tarjan's [7] (relying on the fact that groups of order n have a minimal generating set of size $O(\log(n))$) as well as more sophisticated recent approaches (see for example Rosenbaum's [10]). Put otherwise, in spite of decades of intense research even in the realm of quantum algorithms, no new approach has a worst case complexity that significantly improved upon Tarjan's bound. It is unclear whether this is because of the lack of suitable group invariants or the understanding of the effectiveness of known invariants.

Investigating the descriptive complexity of finite groups in a similar way to graphs could come to aid for the latter. A first attempt to this can be found in recent work by Brachter and Schweitzer [3], where the concepts of *Weisfeiler-Leman equivalences* and *Weisfeiler-Leman dimension* for groups are defined in an analogous manner to graphs. In the context of the descriptive complexity of graphs, the Weisfeiler-Leman (WL) algorithms have proven to be a useful tool. Loosely speaking, for each natural number k, the k-dimensional WL algorithm iteratively refines the colouring of k-tuples of vertices of a graph into isomorphism types until reaching a stable colouring which can be seen as an effective isomorphism invariant. A graph Γ is said to have *WL dimension* at most k, if the invariant computed by the k-dimensional WL algorithm distinguishes Γ from any non-isomorphic graph. While important classes of graphs are known to have bounded WL dimension (graphs with bounded treewidth or genus,

[1]Email: anuj.dawar@cl.cam.ac.uk
[2]Email: dvagnozz@ed.ac.uk

for example [6]), a seminal construction by Cai, Fürer and Immerman [4] has shown that the maximum WL dimension of graphs on n vertices increases linearly with n; whence, there is no fixed k for which the colouring computed by the k-dimensional WL algorithm is a complete invariant. The distinguishing power of the k-dimensional WL algorithm is known to coincide with that of counting logic formulae with at most $k+1$ variables. Counting logics are the extension of first order logic with quantifiers specifying the existence of a certain number of elements satisfying a given formula. Thus, the result by Cai, Fürer and Immerman shattered all hopes of an efficient graph isomorphism test purely based on combinatorial heuristics.

In [3], the authors introduce the WL method for computing isomorphism invariant colourings of the set of k-tuples of group elements and thus define the *WL dimension* of finite groups in a similar fashion to graphs. Whilst an upper bound of $O(\log(n))$ to the WL dimension of groups of order n follows from the definition of the algorithms in themselves, it is still unclear whether this is asymptotically optimal. By embedding graph structures into groups, the authors of [3] attempt to give a construction á la Cai-Fürer-Immerman for finite groups. In this text, we suggest a different approach, by showing that there is a reversible embedding of groups into a class of graphs preserving the WL dimension.

It is well known that group isomorphism is polynomial time reducible to graph isomorphism - see for example [8]. Our main result is that group isomorphism is polynomial time equivalent to the isomorphism problem on Latin square graphs arising from Cayley tables of finite groups and that in both directions the reduction is definable in first order logic. The technical name for a reduction defined in a logic is an *interpretation*, so the result can be formally stated as follows.

Theorem 1. *There exist first-order interpretations $\mathcal{I}$ and $\mathcal{J}(\vec{\mathsf{z}})$, where $\vec{\mathsf{z}}$ is a triple of vertex variables, such that the following hold. Let G be a finite group of order $n \geq 5$ and let Λ be the Latin square graph corresponding to the Cayley table of G. Let V be the vertex set of Λ and let $\vec{v} \in V^3$ be a tuple of pairwise distinct vertices whose induced subgraph in Λ has exactly 2 edges. Then $G^{\mathcal{I}}$ is isomorphic to Λ and $\Lambda^{\mathcal{J}(\vec{\mathsf{z}} \mapsto \vec{v})}$ is a group isomorphic to G.*

Because of the connection between the WL algorithms and counting logics, the following is an immediate corollary.

Corollary 2. *The WL dimension of finite groups of order n is $O(f(n))$ for some function f if, and only if, that of Latin square graphs of groups' Cayley tables is $O(f(n^2))$.*

Latin square graphs are a well studied subclass of *strongly regular graphs*. A graph is said to be *strongly regular* with parameters (n, d, λ, μ) if it has n vertices, it is d regular and any pair of adjacent (rsp. non-adjacent) vertices have exactly λ (rsp. μ) common neighbours. Strongly regular graphs are widely thought to provide hard instances for the graph isomorphism problem. Nevertheless, whilst the idea behind the Cai-Fürer-Immerman constructions has been used to find hard instances of graph isomorphism for methods other than WL, none of these turn out to be classes of strongly regular graphs. There is also a currently open conjecture by Emms et al. [5] claiming that an invariant arising from discrete time quantum walks distinguishes any pair of non-isomorphic strongly regular graphs. Not only this invariant is computable in polynomial time, but its distinguishing power is bounded by that of the 6-dimensional WL algorithm. Should the conjecture be true, this would then imply that the WL dimension of strongly regular graphs is bounded, and by Corollary 2, so would that of finite groups! While there is no concrete evidence as to whether group isomorphism can be decided in polynomial time, it would be surprising if such a task could be achieved simply by combinatorial heuristics.

Structure of the text. Sections 2 and 3 contain background knowledge on logic, the WL algorithms and Latin square graphs. Sections 4 and 5 contain the sketches of the proofs of Theorem 1 and Corollary 2.

2 Logics and algorithms

We view graphs as structures over a vocabulary ($\mathtt{E}$), where $\mathtt{E}$ is a binary relational symbol representing the edge relation. Groups are viewed as structures over a vocabulary ($\mathtt{prod}$) where $\mathtt{prod}$ is a ternary relational symbol representing the group muliplication. That is, for a group G and $g, h, l \in G$ we have that $G \models \mathtt{prod}(g, h, l)$ if, and only if, $gh = l$.

We define a colouring of a set V to be a map $f : V \to X$, where X is a set of colours. For each $x \in X$, the pre-image $f^{-1}(x)$ is said to be a *colour class*.

Counting logic. We assume the reader is familiar with the syntax and semantics of first order logic. By *counting logic* we mean the extension of first order logic by a quantifier $\exists^m$ for each natural number m. If ϕ is a formula of counting logic and $\mathtt{x}$ a vertex variable, then $\exists^m \mathtt{x}.\phi$ is also a counting logic formula, satisfied by a structure $\mathfrak{A}$ (over the corresponding vocabulary) if, and only if, there are at least m elements u for which $(\mathfrak{A}, \mathtt{x} \mapsto u) \models \phi$.

The Weisfeiler-Leman method for graphs. Given a graph Γ with vertex set V and an integer $k \geq 2$, the k-dimensional WL algorithm iteratively refines the colouring of V^k into isomorphism types according to the connectivity properties of the entries of the k-tuple within Γ. Let α_0 be a canonical colouring[3] of V^k into the isomorphism types of Γ; that is, for all $\vec{u}, \vec{v} \in V^k$, $\alpha_0(\vec{u}) = \alpha_0(\vec{v})$ if, and only if the map $u_i \mapsto v_i$ for each $1 \leq i \leq k$ is a local isomorphism. This colouring is refined as follows. Let α_i be the colouring of V^k after i iterations. Then for each $\vec{u} \in V^k$

$$\alpha_{i+1}(\vec{u}) = (\alpha_i(\vec{u}), \{\{(\alpha_i(\vec{u}_{1 \leftarrow x}), \alpha_i(\vec{u}_{2 \leftarrow x}), \ldots, \alpha_i(\vec{u}_{k \leftarrow x})) \mid x \in V\}\})$$

where $\{\{\ldots\}\}$ indicates a multiset and for each i and $x \in V$, $\vec{u}_{i \leftarrow x}$ is the element of V^k obtained by replacing the ith entry of $\vec{u}$ by x. Since V is assumed to be finite, there exist an integer s, such that the partitions induced by the colourings α_j for $j \geq s$ are the same (that is, the partitions stabilise and do not refine any further). For the minimal such s, the colouring α_s is the output of the algorithm on input Γ. It is easy to see that since α_0 is chosen to be canonical, α_s is isomorphism invariant. A pair of graphs are then said to be *k-WL equivalent* if they both yield the same output colouring from the k-dimensional WL algorithm. The *WL dimension* of a graph Γ is defined to be the least integer k for which any graph Γ' is k-WL equivalent to Γ if, and only if, it is isomorphic to Γ. For a class of graphs $\mathcal{G}$, we define its WL dimension to be the maximal of the WL dimensions of all graphs in $\mathcal{G}$ if such a maximum exists, otherwise we say that the WL dimension of $\mathcal{G}$ is unbounded. A nice argument using the Immerman-Lander pebble game is used in [4] to show that two graphs are k-WL equivalent if, and only if, no counting logic formula with at most $k + 1$ variables can distinguish them.

The Weisfeiler-Leman method for groups. As explained in Section 3 of [3], there are several natural ways of defining the WL equivalences on groups, all of which lead to the same notion of equivalence in a well defined sense. For the purpose of this text we choose the one dubbed as the *algebraic* WL equivalences in [11]. In a similar way to graphs, given a group G and an integer $k \geq 2$, the k-dimensional algebraic WL algorithm iteratively refines a canonical colouring of G^k into isomorphism types until reaching a colouring that is not refined any further. The initial canonical colouring α_0 is such that for all $\vec{g}, \vec{h} \in G^k$, $\alpha_0(\vec{g}) = \alpha_0(\vec{h})$ if, and only if, the map $g_i \mapsto h_i$ for each $1 \leq i \leq k$ extends to an isomorphism between the subgroups $\langle g_1, \ldots, g_k \rangle$ and $\langle h_1, \ldots, h_k \rangle$ of G. This colouring is then refined using the same iterative procedure in the case for graphs until reaching a colouring α_s whose induced partition of G^k is not refined. The colouring α_s is then the invariant assigned to the group. We define the notions of two groups being k-dimensional WL equivalent and the WL dimensions of a group

[3] By *canonical*, we mean that if α and α' are such colourings for graphs Γ and Γ' respectively, then $\alpha(\vec{u}) = \alpha'(\vec{v})$ if, and only if, the map $u_i \mapsto v_i$ for each $1 \leq i \leq k$ is a partial isomorphism from Γ to Γ'.

and a class of groups similarly to graphs. Also in an analogous way to graphs, one can show that the distinguishing power of the k-dimensional WL algorithms on groups coincides with that of counting logic formulae with at most $k+1$ variables.

3 Latin square graphs: symmetries and geometry

A *Latin square* over a finite alphabet A is an $A \times A$ array with entries from A such that every element of A occurs exactly once in each row and each column. We may represent a Latin square as a ternary relation $L \subset A^3$ where $(a,b,c) \in L$ if, and only if, the entry in row a and column b is c. To a Latin square L, we associate a graph Λ_L whose vertex set is A^2 and vertices (a,b) and (a',b') are connected by an edge if, and only if, $a = a'$, or $b = b'$, or if for some $c \in A$ it holds that $(a,b,c),(a',b',c) \in L$. Put otherwise, the vertex set of Λ_L is the set of cells of the Latin square, and two cells share an edge if, and only if, they are in the same row or in the same column, or they have the same entry. We call Λ_L the *Latin square graph* arising from L. For a Latin square over an alphabet of size n, the corresponding Latin square graph has size n^2. It can be shown that a Latin square graph of size n^2 is strongly regular with parameters $(n^2, 3(n-1), n, 6)$.

In general, Latin squares can be viewed as multiplication tables of *quasigroups*. A quasigroup is a set A with a multiplication $\circ$ such that for any $a,b \in A$, the equations $a \circ x = b$ and $y \circ a = b$ have a unique solution in x and y. It is then easy to see that Cayley tables of groups are in fact Latin squares. The following is a fundamental result by Albert [1].

Theorem 3. *Groups G and H are isomorphic if, and only if, the Latin square graphs Λ_G and Λ_H arising from their Cayley tables are isomorphic.*

Let us now look at the geometry of Latin square graphs. Recall that a *k-clique* of a graph is a set of vertices whose induced subgraph is a complete graph. A clique in a graph Γ is said to be *maximum* if there are no larger cliques in Γ. Let L be a Latin square over an alphabet of size n and let Λ_L be the corresponding Latin square graph. A simple combinatorial argument can be used to show that if $n \geq 5$, the maximum cliques are of size n, and each pair of adjacent vertices is contained in a unique maximum clique; furthermore, every 5-clique is contained in a unique maximum clique. From this, we deduce that the first order formula

$$\mathtt{clq}(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3) = \mathtt{E}(\mathtt{x}_1,\mathtt{x}_2) \wedge (\mathtt{x}_1 = \mathtt{x}_3 \vee \mathtt{x}_2 = \mathtt{x}_3) \vee \exists \mathtt{x}_4 \mathtt{x}_5 . \Big[\bigwedge_{i,j \in [5], i \neq j} \mathtt{E}(\mathtt{x}_i, \mathtt{x}_j) \Big]. \tag{1}$$

defines the maximum cliques of Λ_L in that for all adjacent vertices u, v, $(\Lambda_L, \mathtt{x}_1 \mapsto u, \mathtt{x}_2 \mapsto v)$ satisfies $\mathtt{clq}(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3)$ whenever $\mathtt{x}_3$ is interpreted by a vertex in the unique maximum clique containing u and v. The maximum cliques of Latin square graphs of size $n^2 \geq 25 = 5^2$ coincide with the cells in a row or a column or with cells containing a same entry from the underlying alphabet. Thus, any two maximum cliques are either disjoint or intersect in a single vertex. These pairs of maximum cliques are referred to as being *parallel* and *orthogonal* respectively.

The automorphism groups of Latin square graphs arising from Cayley tables of groups have been well studied and characterised in [2]. For the purpose of the present text, all we need is that for Latin square graphs arising from Cayley tables of groups, the automorphism group acts transitively; furthermore, its induced action on maximum cliques is *almost* 2-transitive in that it is transitive on the set of parallel pairs of cliques and on the set of orthogonal pairs of cliques (see Section 4.5.3 in [11] for a proof).

4 Proof sketch of Theorem 1

In the formal statement of Theorem 1 we mentioned the notion of an interpretation. We shall not use the formal definition, but the following intuitive idea should suffice. Let $\sigma = (\mathtt{R}_1, \ldots, \mathtt{R}_m)$ and

$\tau = (\mathtt{T}_1, \ldots, \mathtt{T}_l)$ be relational vocabularies. A $[\sigma, \tau]$-interpretation $\mathcal{I}(\vec{\mathtt{z}})$ is a mapping taking a σ-structure $\mathfrak{A}$ to τ-structure $\mathfrak{A}^{\mathcal{I}(\vec{\mathtt{z}})}$. The map is dictated by a sequence of formulae from some logic and the values interpreting the parameter variables $\vec{\mathtt{z}} = (\mathtt{z}_1, \ldots, \mathtt{z}_p)$. Every interpretation has at least two formulae: the *domain formula* ϕ_D and the *equivalence formula* ϕ_E. Together, these define the domain of $\mathfrak{A}^{\mathcal{I}(\vec{\mathtt{z}})}$ as the set of equivalence classes on a set of r-tuples of the domain of $\mathfrak{A}$, for some r. For each relational symbol $\mathtt{T}_i$ the interpretation has a formula $\phi_{\mathtt{T}_i}$ defining the relation $\mathtt{T}_i$ interpreted in $\mathfrak{A}^{\mathcal{I}(\vec{\mathtt{z}})}$. Each of the formulae ϕ_D, ϕ_E and $\phi_{\mathtt{T}_i}$ is written in the vocabulary σ and may contain the variables in $\vec{\mathtt{z}}$ as free variables. The main result we will be using regarding interpretations is sometimes referred to as the *Lemma for Syntactical Interpretations* (the proof of which can be found in in [9]). Informally, it claims that τ-structures $\mathfrak{A}^{\mathcal{I}(\vec{\mathtt{z}})}$ and $\mathfrak{B}^{\mathcal{I}(\vec{\mathtt{z}})}$ both satisfy a formula ψ if, and only if, σ-structures $\mathfrak{A}$ and $\mathfrak{B}$ both satisfy a formula $\psi^{-\mathcal{I}(\vec{\mathtt{z}})}$ obtained by replacing occurrences of $\mathtt{T}_i$ by $\phi_{\mathtt{T}_i}$, with the appropriate variables. The reader may note that $\psi^{-\mathcal{I}(\vec{\mathtt{z}})}$ written as such is a formula written in the vocabulary σ. Rather than formally defining the interpretations, we describe the construction of the Latin square graph from the group and vice-versa, and the reader can verify that each set in the construction is first-order definable. Let G be a finite group and Λ the Latin square graph arising from its Cayley table.

4.1 Construction of $\mathcal{I}$: from groups to Latin square graphs

The vertex set of a Λ can be identified with G^2 so the domain formula of $\mathcal{I}$ defines G^2 and the equivalence formula is the trivial notion of equality in G^2. From the definition of a Latin square graph, the edge relation of Λ is given by the set of pairs of elements $(a, b), (a', b') \in G^2$ such that $a = a'$ or $b = b'$ or $ab = a'b'$. This can be clearly defined using the equality relation $=$ and the ternary relation $\mathtt{prod}$ representing the product relation in G.

4.2 Construction of $\mathcal{J}(\vec{\mathtt{z}})$: from Latin square graphs to groups.

We now sketch the construction of $\mathcal{J}(\vec{\mathtt{z}})$. The aim is to ensure that $H = \Lambda^{\mathcal{J}(\vec{\mathtt{z}} \mapsto \vec{v})}$ is a structure isomorphic to the group G. *A priori*, H is a ternary relational structure with the product relation of a quasigroup. To show that it is indeed the product relation of a group one may use the *translational property* of Cayley tables; we refer the reader to Section 4.5.2 of [11] for details about this.

Recall that the assumption $n \geq 5$ in Theorem 1 means that the elements of G are in bijection with those of any maximum clique of Λ. Note that if $\vec{v} \in V^3$ satisfies the assumptions in Theorem 1, then there are two orthogonal canonical maximum cliques which are a candidate for the universe of H. Indeed, since $\{v_1, v_2, v_3\}$ induce a subgraph of Λ with exactly 2 edges, we may assume that v_1 is adjacent to both v_2 and v_3, and since Λ has at least 25 vertices, from the discussion in Section 3, there is a unique pair of orthogonal cliques R, C, such that $\{v_1, v_2\} \subset R$ and $\{v_2, v_3\} \subset C$. Since the automorphism group of Λ is transitive on pairs of orthogonal cliques, fixing $\vec{v}$, and therefore R and C, can be thought of as fixing which cliques correspond to rows and which to columns. Thus, we may view R as a row and C as a column of the multiplication table of H and we may take R to be the universe without loss of generality. Now, for any pair of othogonal maximum cliques there is exactly one maximum clique orthogonal to both of these containing a given vertex $v \in V$. Given any $g \in R$, this property allows us to determine which vertex $g^{(C)}$ of C corresponds to a cell with entry g. For $h \in R$, let gh denote the vertex in the intersection of the (unique) maximum clique parallel to R containing $g^{(C)}$ with the (unique) maximum clique parallel to C containing h. The entry of the cell corresponding to this vertex must be that of the unique vertex k in the intersection of R with the unique clique orthogonal to both R and C containing the vertex gh. The product relation of H as a ternary relation on R is then the set of all (g, h, k) thus constructed. It is apparent from the above that the main formulae for the construction of $\mathcal{J}(\vec{\mathtt{z}})$ rely on the definability of the following subsets of V, all of which, the reader may verify, are first-order definable:

1. The maximum clique containing a given pair of adjacent vertices of Λ.

2. The maximum clique orthogonal to a pair of orthogonal cliques and containing a given vertex.
3. The maximum clique parallel to a given maximum clique and containing a given vertex.

5 Proof of Corollary 2 and conclusions

Corollary 2 follows from the connection between the WL algorithms and counting logic outlined in Section 2, and applying the Lemma for Syntactical Interpretations. Whilst the main side effect of our main results are the consequences of the truthfulness of the conjecture by Emms et al., there are further research directions to consider. Importantly, a recent result by Fuhlbrück et al. characterises the graphs with WL dimension larger than 2 with colour classes of size at most 4, reaching the conclusion that the Cai-Fürer-Immerman constructions naturally occur in this setting and are not *ad hoc*. Generalising this characterisation to graphs with larger colour classes and WL dimension would be an important step towards understanding where do Latin square graphs place themselves in terms of combinatorial heuristics. In particular, it is worth investigating whether Latin square graphs with bounded size colour classes have bounded WL dimension.

References

[1] A. A. Albert. "Quasigroups. I". In: *Transactions of the American Mathematical Society* 54.3 (1943), pp. 507–519.

[2] R. A. Bailey, P. J. Cameron, C. E. Praeger, and C. Schneider. *The geometry of diagonal groups*. Preprint. 2020. DOI: 10.48550/ARXIV.2007.10726.

[3] J. Brachter and P. Schweitzer. "On the Weisfeiler-Leman Dimension of Finite Groups". In: *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS '20. New York, NY, USA: Association for Computing Machinery, 2020, pp. 287–300. DOI: 10.1145/3373718.3394786.

[4] J. Y. Cai, M. Fürer, and N. Immerman. "An optimal lower bound on the number of variables for graph identification". In: *Combinatorica* 12.4 (1992), pp. 389–410.

[5] D. Emms, E. R. Hancock, S. Severini, and R. C. Wilson. "A matrix representation of graphs and its spectrum as a graph invariant." eng. In: *The Electronic Journal of Combinatorics [electronic only]* 13.1 (2006), Research paper R34, 14 p.–Research paper R34, 14 p.

[6] M. Grohe. *Descriptive Complexity, Canonisation, and Definable Graph Structure Theory*. Lecture Notes in Logic. Cambridge University Press, 2017. DOI: 10.1017/9781139028868.

[7] G. L. Miller. "On the $n^{\log n}$ Isomorphism Technique (A Preliminary Report)". In: *Proceedings of the Tenth Annual ACM Symposium on Theory of Computing*. STOC '78. New York, NY, USA: Association for Computing Machinery, 1978, pp. 51–58. DOI: 10.1145/800133.804331.

[8] G. L. Miller. "Graph isomorphism, general remarks". In: *Journal of Computer and System Sciences* 18.2 (1979), pp. 128–142. DOI: https://doi.org/10.1016/0022-0000(79)90043-6.

[9] M. Otto. "Bounded Variable Logics and Counting: A Study in Finite Models". In: Lecture Notes in Logic. Cambridge University Press, 2017.

[10] D. J. Rosenbaum. *Bidirectional Collision Detection and Faster Deterministic Isomorphism Testing*. Preprint. 2013. DOI: 10.48550/ARXIV.1304.3935.

[11] D. Vagnozzi. "Variations on the theme of higher dimensional Weisfeiler-Leman algorithms". PhD thesis. University of Cambridge, 2022.

On vertex bisection width of random d-regular graphs

Josep Díaz[1*,§], Öznur Yaşar Diner[2†,‡], Maria Serna[3*,§], Oriol Serra[4*,§]

[*]ALBCOM Research Group, Computer Science Department. Universitat Politècnica de Catalunya, Barcelona, Spain
[†]Computer Engineering Department, Kadir Has University, Istanbul, Turkey
[‡]Mathematics Department. Universitat Politècnica de Catalunya, Barcelona, Spain
[§]Institut de Matemàtiques de la UPC-BarcelonaTech (IMTech), Universitat Politècnica de Catalunya, Barcelona, Spain

This research is supported by the Spanish Agencia Estatal de Investigación*, J. D. and M. S. under grant PID2020-112581GB-C21 and Ö. Y. D and O. S. under grant PID2020-113082GB-I00.*

Abstract

Vertex Bisection is a graph partitioning problem in which the aim is to find a bisection that minimizes the number of vertices in one partite set that has a neighbor in the other set. Here we are interested in giving upper bounds on the vertex bisection width of random d-regular graphs. Our approach is based on analyzing a greedy algorithm by using the Differential Equations Method. In this way, we obtain the first known upper bounds for the vertex bisection width in random regular graphs. The results are compared with experimental ones and with lower bounds obtained by Kolesnik and Wormald (Lower Bounds for the Isoperimetric Numbers of Random Regular Graphs, SIAM J. on Disc. Math. 28(1), 553-575, 2014) for this parameter.

1 Introduction

Partitioning the vertices of a given graph G into subsets, subject to certain constraints, is an important algorithmic problem, which is used to model a large range of practical applications, see e.g. [2]. The theoretical study of graph partitioning problems has also arisen a great range of interest.

Minimum edge bisection is a graph partitioning problem that has received considerable attention from both the theoretical and practical aspects, as the problem measures the amount of information that can flow through a network.

Let $G = (V, E)$ be a graph with n vertices. Let $(S, V \setminus S)$ be a partition of the vertex set V into equal sized sets when n is even and differing by one when n is odd. Such a partition is called a *bisection*. The *(edge) width* of a bisection $(S, V \setminus S)$ is the number of edges connecting the parts S and $V \setminus S$, that is, $bw((S, V \setminus S)) = |\{uv \in E : u \in S, v \in V \setminus S\}|$. The minimum edge bisection or *bisection width* of G is the minimum width over all bisections,

$$bw(G) = \min\{bw((S, V \setminus S)) : (S, V \setminus S) \text{ is a bisection}\}.$$

The corresponding decision problem, Minimum edge bisection, has been a very productive research area, using theoretical and empirical methods. Concerning its algorithmic complexity Minimum edge bisection is NP-complete [15] for general graphs and it remains in NP-complete when restricted to

[1]Email: diaz@cs.upc.edu.
[2]Email: oznur.yasar@khas.edu.tr.
[3]Email: mjserna@cs.upc.edu.
[4]Email: oriol.serra@upc.edu.

many graph classes, in particular d-regular graphs even for $d = 3$ [6]. The problem is efficiently solvable for trees [26], hypercubes [25], grid graphs with a constant number of holes [13] and on graphs with bounded treewidth [17]. Regarding poly-time approximation algorithms, it is known that there exists no polynomial time approximation scheme to approximate bisection width within a constant factor [20]. The best approximation algorithm for the problem, achieves a ratio of $O(logn)$ [27]. From the practical point of view, there have been a slow improvement of efficient heuristics, from the simulate-annealing in the 1970's [19] to the one that in practice is very efficient [9].

A fruitful line of research has been the study of asymptotic tight bounds for the bisection width of random d-regular graphs, in particular for $d = 3$. Buser showed the existence of a cubic graph with G with $bw(G) > \frac{n}{256}$ [7]. Bollobás extended Buser's result to show that for almost all cubic graphs G, $bw(G) > \frac{n}{11}$, and extend the lower bound for d-regular graphs [4]. Clark and Entringer showed that for almost all cubic graphs, $bw(G) \leq \frac{(n+138)}{3}$. They also proved that almost every d-regular G $c_d n \leq bw(G)$ where $c_d \to d/4$ as $d \to \infty$, which is asymptotically correct [8]. In the last 30 years, several improvements have tightened the bounds for bisection width. Alon [1] give upper bounds for the edge bisection width for general values of the degree d. Note that the method in [1] can be turned into a randomized algorithm. These bounds were recently improved by Lyons [24]. Lichev and Mitsche [23] obtained very tight upper and lower bounds for the cubic case. Díaz, Serna and Wormald [11] gave a randomized greedy algorithm for constructing bisections with small edge width which provides, for $4 \leq d \leq 12$, the best known upper bounds for the edge–bisection of random d–regular graphs. In that paper, the algorithm is analyzed by means of the Differential Equation Method (DEM). The DEM is introduced by Kurtz [22] and was used in various combinatorial problems [18]. A general frame work to use the DEM to obtain a greedy algorithm that would create a semi-martingale that is concentrated is given by Wormald [29]. In this paper we adapt the same strategy to obtain bounds on the *vertex bisection width* of random d-regular graphs.

In the vertex bisection problem the aim is to minimize the number of vertices in S that are connected by an edge to some vertex in $V \setminus S$. Thus the *(vertex) width* of a bisection $(S, V \setminus S)$ is $vbw((S, V \setminus S)) = |\{u \in S : \exists v \in V \setminus S, uv \in E\}|$. The *vertex bisection width* of G is the minimum width over all bisections,

$$vbw(G) = \min\{vbw((S, V \setminus S)) : (S, V \setminus S) \text{ is a bisection}\}.$$

In spite that the Min Vertex Bisection arises in important applications, as the study of message passing and fault tolerance in interconnection networks, see e.g. [12, 28], the problem has received little attention, in particular from the theoretical point of view. The problem is known to be NP-complete for general graphs and it is solvable in polynomial time for hypercubes [5]. There have been heuristics based in integer linear programming and branch and bound [14, 16], where due to the exhaustive search nature of the techniques, their applicability is reduced only to graphs of small size.

Recently, Kolesnik and Wormald [21] give asymptotically almost sure lower bounds for the vertex bisection width of random d–regular graphs for every $d \geq 3$. The main motivation of this paper is to provide corresponding upper bounds by producing bisections with small width. This is achieved by proposing a randomized greedy algorithm which is suited for the vertex bisection problem, and analyze it by means of the Differential Equation Method.

2 A greedy algorithm and the Differential Equation Method

We use the pairing model to generate random d–regular graphs with n vertices, where nd is assumed to be even. The pairing model chooses a random matching on a set of nd points that are grouped into bags of size d, each bag corresponding to a vertex. A random d–regular graph can be obtained by a graph process $(G_t, t = 0, 1, \ldots, nd)$ by arbitrarily choosing an unpaired point and by pairing it with a randomly chosen point, exposing in this way an edge of the graph, at time t (see e.g. [29].)

The *differential equation method* is a set of techniques used in combinatorics and algorithmics to analyze dynamic random processes that evolve in tiny steps (with respect to the final structure). For example, one step can be exposing a new edge in the construction of a large graph. The method produces tight bounds on a set of random variables that almost asymptotically surely hold at every step. The very useful characteristic of the method is that, at each step, each random variable is concentrated around its expectation as it evolves from step to step, see [30, Theorem 5.1] and [10].

Upper bounds for $vbw(G)$ are obtained by exhibiting vertex bisections, or two–colorings, (R, B) with a set R_0 of interior vertices of color red as large as possible. For $d \geq 4$ it is likely that the partitions which minimize the edge width and the ones minimizing the vertex width have different structural properties. In cases where both of these parameters are known, such as the hypercube, the edge bisection is minimized by matchings while vertex bisection is minized by balls. This suggests the following algorithm to produce suitable vertex bisections with large $|R_0|$. For a vertex x_0 and integer r we denote the set of vertices at graph distance at most r from x_0 as $B(x_0, r)$.

Algorithm: Create a vertex bisection
Choose a random vertex x_0 and set $r = 0$
while $|B(x_0, r)| < n/2$ **do**
  $r \to r + 1$
  Set $r_0 = r - 1$, $R = B(x_0, r_0)$
while $|R| < n/2$ **do**
  select $x \in B(x_0, r_0)$ with $d/2 - 1 < |B(x, 1) \cap R| < d$
  $R \to R \cup (B(x, 1) \cap V \setminus R)$
Return $(R, V \setminus R)$.

In words, we first start by coloring red all vertices at graph distance at most r_0 from a vertex x_0, where r_0 is the largest integer for which the ball centered at x_0 and radius r_0 has cardinality at most $n/2$. In particular, $R_0 \supseteq B(x_0, r_0 - 1)$. Notice that this is not a bisection yet. We next proceed to enlarge R up to $n/2$ by coloring red the uncolored vertices adjacent to a given red vertex, thus increasing at least by one the cardinality of R_0. We do so for vertices with less than $d/2$ uncolored neighbours to minimize the increase of vertices in the bisection. The algorithm will seemingly produce a partition with small bisection in random regular graphs. The reason is that, while $r < r_0$, the balls $B(x_0, r)$ expand exponentially in r. Moreover, the distribution of the degrees in $B(x_0, r_0)$ makes it possible, for this radius, to complete the red part of the bisection by adding vertices with at most $d/2$ uncolored neighbors. A probabilistic argument following the lines of Bollobás and Fernández de la Vega [3] explains why this approach is feasible.

Let us briefly explain how the differential equation method is used to analyze the above algorithm.

We construct the d–regular graph along the configuration model process. Every vertex is a bag with d points. Along the process, points are selected one by one in an arbitrary manner and, when a point is selected, its mate is chosen uniformly at random among the unselected points. This exposes one edge of the configuration joining the vertices containing the two paired points.

The algorithm runs through this configuration process in several phases. At each step an unmatched point is selected in a red vertex, with a rule depending on the phase of the algorithm; a second point is chosen uniformly at random among the unmatched points, it is paired with the former point and the corresponding edge is exposed.

In order to apply the Differential Equation Method to estimate the vertex bisection in the output of the algorithm, we define the random variables

- R_i, $0 \leq i \leq d$, counts the number of red vertices with i unmatched points.
- $nR = \sum_{i=0}^{d} R_i$, counts the number of red vertices.
- $pR = \sum_{i=0}^{d} iR_i$, counts the number of unmatched points in vertices colored red.

- Z_i, $1 \leq i \leq d$, counts the number of non red vertices with i unmatched points.
- $nZ = \sum_{i=0}^{d} Z_i$ counts the number of non red vertices.
- $pZ = \sum_{i=0}^{d} iZ_i$ counts the number of unmatched points in verttices not colored red.
- $pW = pR + pW$, counts the total number of unmatched points.

All the variables depend on the step t of the algorithm, but this dependence is not made explicit in the notation. At the end of the algorithm, the number of red vertices in the bisection is

$$|R| - |R_0|,$$

which we take as an upper bound for the bisection width.

We initialize the algorithm by coloring red a ball of a sufficiently small radius so that we have a complete d–ary tree of some depth k with $1 + d\sum_{i=1}^{k-1}(d-1)^i$ vertices, of which $d(d-1)^k$ of them have $(d-1)$ neighbours outside the ball. This provides an initialization of the variables R_i and Z_i in the first phase.

We then run the algorithm till either the variables R_i are all zero for $i \geq 1$ (no additional unmatched red points are available) or $|R| = n/2$ (the last phase). In the first case we run a next phase of the algorithm by declaring red all vertices counted with less than d unmatched points and initializing the variables correspondingly.

Within each phase, at each step t we compute the expected changes in the values of the variables (up to $O(1/nW)$ terms). For instance, for the variables R_i, $0 < i < d$, we have

$$\begin{aligned}\Delta(R_i) = &-2\frac{(iR_i)^2}{(pR)(nW)}\\ &-\frac{iR_i(nW - iR_i - (i+1)R_{i+1})}{(pR)(nW)} - \frac{(pR - iR_i - (i+1)R_{i+1})(iR_i)}{(pR)(nW)}\\ &+\frac{(i+1)R_{i+1}(nW - iR_i - (i+1)R_{i+1}}{(pR)(nW)} + \frac{(pR - iR_i - (i+1)R_{i+1})((i+1)R_{i+1}}{(pR)(nW)}\\ &+2\frac{((i+1)R_{i+1})^2}{(pR)(nW)},\end{aligned}$$

monitoring the two random choices for the points to be paired, the first one among points counted by rP and the second one among all nW available points. The variable R_i is modified if the points are chosen among the ones counted by R_i (decreasing its value) or R_{i+1} (increasing its value). Similar expressions are obtained for the expected changes in the values of the variables ΔZ_i and $\Delta R_0, \Delta Z_d$.

The differential equations method is based on translating these incremental expected changes into a system of differential equations by turning the steps into a continuous time. The solutions of this system evolve closely to their discrete counterparts for large n. The variable n, conveniently normalized, is converted to a continuous time t for the system of differential equations. By an additional argument on concentration of the values on their expected values, the main result in [30, Theorem 5.1] shows that the values of the solutions at the stopping time (in our case when we obtain $n/2$ red vertices) provides an asymptotically good approximation of the values of the variables on the output of the algorithm, from which we can compute the resulting vertex bisection width given by $n/2 - R_0$, as R_0 counts the number of interior red vertices in the partition. The applicability of [30, Theorem 5.1] requires simply that the variables in the process have bounded variation (in our case these variations are bounded by two in absolute value) and that the corresponding continuous versions are Lipschitz (in our case they are quadratic functions of the variables) together with the natural integrability conditions of the system in the domain of interest.

Several phases of the algorithm governed by the same system of differential equations are run where each phase stops when there are no unmatched red points left. These results are feeding the initial

values of the variables in the next phase, until the final phase, which stops at $nR = n/2$. In order to initialize the whole process, we color the vertices of a ball of small radius with red, giving nR, a small fraction of n, where only the variables R_0, R_{d-1}, Z_1 and Z_0 are different from 0.

In order to assess the the results, Table 1 displays the average of the experimental results obtained by running a randomized greedy algorithm (that uses a priotrized list of vertices and a majority rule similar to [11]) and the greedy algorithm defined here. Experiments are run on several instances of random regular graphs with $n = 10^5$ vertices provided by the Python package NetworkX-2.5. For comparison, Table 1 includes the lower bounds reported in [21] and the upper bounds obtained from the DEM detailed above (with the RungeKutta4 program in SciPy library.) In all of the cases the displayed values give the proportion α of vertices in the red part which belong to the bisection, so that the value of the bisection is $\alpha(n/2)$.

Degree d	4	5	6	7	8	9	10
DEM Upper Bound	0.60301	0.78493	0.82558	0.88827	0.94447	0.96242	0.97957
Experiments (Randomized)	0.57816	0.75624	0.81718	0.88686	0.93160	0.95840	0.97480
Experiments (Balls Method)	0.52531	0.6646	0.69045	0.76240	0.78082	0.80040	0.85026
Lower Bound [21]	0.28966	0.40859	0.50190	0.57466	0.63178	0.67716	0.71371

Figure 1: Results for d–regular graphs with $d = 4, \ldots, 10$.

3 Conclusions

For some graph classes the bisections which provide the minimum values of edge bisection also minimize the vertex bisection. In particular, for the case of cubic graphs we do not report our results since they indicate that in this case the edge bisection and vertex bisection are very close to each other and the best known bounds in [23] are significant for vertex bisection.

The method given here to estimate upper bounds for the minimum vertex bisection width of random d–regular graphs can be applied to a range of meaningful degrees, say from $4 \leq d \leq 10$. The results show a considerable gap between the lower and upper bounds of the vertex bisection. We believe that there is more room for improvement in the lower bound than in the upper bound. We tried different algorithms for obtaining good vertex bisections and the one proposed here is the best performing ones.

For larger values of d the lower bounds in [21] come close to $n/2$, the trivial upper bound for the minimum vertex bisection width. For example, for $d = 100$ the lower bound is $0.9785(n/2)$. This is also close to the expected value of a random bisection, showing very small variability among distinct bisections. We note that the expected vertex bisection of a random partition into two equal parts is $n/2(1 - 1/2^d)$ which tends to $n/2$ with $d \to \infty$. Therefore, the problem is meaningful only for a range of small values of the degree.

References

[1] N. Alon. "On the edge-expansion of graphs". In: *Combin. Probab. Comput.* 6.2 (1997), pp. 145–152. DOI: 10.1017/S096354839700299X.

[2] C.-E. Bichot and P. Siarry, eds. *Graph partitioning.* ISTE, London; John Wiley & Sons, Inc., Hoboken, NJ, 2011, pp. xvi+368.

[3] B. Bollobás and W. Fernandez de la Vega. "The diameter of random regular graphs". In: *Combinatorica* 2.2 (1982), pp. 125–134. DOI: 10.1007/BF02579310.

[4] B. Bollobás. "The isoperimetric number of random regular graphs". In: *European J. Combin.* 9.3 (1988), pp. 241–244. DOI: 10.1016/S0195-6698(88)80014-3.

[5] U. Brandes and D. Fleischer. "Vertex bisection is hard, too". In: *J. Graph Algorithms Appl.* 13.2 (2009), pp. 119–131. DOI: 10.7155/jgaa.00179.

[6] T. N. Bui, S. Chaudhuri, F. T. Leighton, and M. Sipser. "Graph bisection algorithms with good average case behavior". In: *Combinatorica* 7.2 (1987), pp. 171–191. DOI: 10.1007/BF02579448.

[7] P. Buser. "On the bipartition of graphs". In: *Discrete Appl. Math.* 9.1 (1984), pp. 105–109. DOI: 10.1016/0166-218X(84)90093-3.

[8] L. H. Clark and R. C. Entringer. "The bisection width of cubic graphs". In: *Bull. Austral. Math. Soc.* 39.3 (1989), pp. 389–396. DOI: 10.1017/S0004972700003300.

[9] D. Delling, A. V. Goldberg, I. Razenshteyn, and R. F. Werneck. "Exact Combinatorial Branch-and-Bound for Graph Bisection". In: ALENEX '12. Kyoto, Japan: Society for Industrial and Applied Mathematics, 2012, pp. 30–44.

[10] J. Díaz and D. Mitsche. "The cook-book approach to the differential equation method". In: *Computer Science Review* 4.3 (2010), pp. 129–151. DOI: https://doi.org/10.1016/j.cosrev.2010.04.002.

[11] J. Díaz, M. J. Serna, and N. C. Wormald. "Bounds on the bisection width for random d-regular graphs". In: *Theoret. Comput. Sci.* 382.2 (2007), pp. 120–130. DOI: 10.1016/j.tcs.2007.03.003.

[12] J. Díaz, J. Petit, and M. Serna. "A Survey of Graph Layout Problems". In: *ACM Comput. Surv.* 34.3 (2002), pp. 313–356. DOI: 10.1145/568522.568523.

[13] A. E. Feldmann and P. Widmayer. "An $\mathcal{O}(n^4)$ time algorithm to compute the bisection width of solid grid graphs". In: *Algorithms—ESA 2011*. Vol. 6942. Lecture Notes in Comput. Sci. Springer, Heidelberg, 2011, pp. 143–154. DOI: 10.1007/978-3-642-23719-5_13.

[14] H. Fraire, J. David Terán-Villanueva, N. C. García, J. J. G. Barbosa, E. R. d. Angel, and Y. G. Rojas. "Exact Methods for the Vertex Bisection Problem". In: *Recent Advances on Hybrid Approaches for Designing Intelligent Systems*. Ed. by O. Castillo, P. Melin, W. Pedrycz, and J. Kacprzyk. Cham: Springer International Publishing, 2014, pp. 567–577. DOI: 10.1007/978-3-319-05170-3_40.

[15] M. R. Garey, D. S. Johnson, and L. Stockmeyer. "Some simplified NP-complete graph problems". In: *Theoret. Comput. Sci.* 1.3 (1976), pp. 237–267. DOI: 10.1016/0304-3975(76)90059-1.

[16] P. Jain, G. Saran, and K. Srivastava. "Branch and bound algorithms for vertex bisection minimization problems". In: *Advanced Computing and Comm. Tech.* 452. Springer, 2016. DOI: 10.1007/978-981-10-1023-1_2.

[17] K. Jansen, M. Karpinski, A. Lingas, and E. Seidel. "Polynomial time approximation schemes for max-bisection on planar and geometric graphs". In: *SIAM J. Comput.* 35.1 (2005), pp. 110–119. DOI: 10.1137/S009753970139567X.

[18] R. M. Karp and M. Sipser. "Maximum matching in sparse random graphs". In: *22nd Annual Symposium on Foundations of Computer Science (sfcs 1981)*. IEEE. 1981, pp. 364–375.

[19] B. W. Kernighan and S. Lin. "An efficient heuristic procedure for partitioning graphs". In: *The Bell System Technical Journal* 49.2 (1970), pp. 291–307. DOI: 10.1002/j.1538-7305.1970.tb01770.x.

[20] S. Khot. "Ruling out PTAS for graph min-bisection, dense k-subgraph, and bipartite clique". In: *In FOCS-04* (2004), pp. 136–145.

[21] B. Kolesnik and N. Wormald. "Lower bounds for the isoperimetric numbers of random regular graphs". In: *SIAM J. Discrete Math.* 28.1 (2014), pp. 553–575. DOI: 10.1137/120891265.

[22] T. G. Kurtz. "Solutions of ordinary differential equations as limits of pure jump Markov processes". In: *J. Appl. Probability* 7 (1970), pp. 49–58. DOI: 10.2307/3212147.

[23] L. Lichev and D. Mitsche. *On the minimum bisection of random* $3-$*regular graphs*. Preprint. 2020. DOI: 10.48550/ARXIV.2009.00598.

[24] R. Lyons. “Factors of IID on trees”. In: *Combin. Probab. Comput.* 26.2 (2017), pp. 285–300. DOI: 10.1017/S096354831600033X.

[25] R. MacGregor. “On Partitioning a Graph: A Theoretical and Empirical Study”. PhD thesis. University of California, Berkeley, 1978.

[26] K. Nakano. “Linear layout of generalized hypercubes”. In: vol. 14. 1. Advances in parallel and distributed computational models (Ft. Lauderdale, FL, 2002). 2003, pp. 137–156. DOI: 10.1142/S0129054103001637.

[27] H. Räcke. “Optimal hierarchical decompositions for congestion minimization in networks”. In: *STOC’08*. ACM, New York, 2008, pp. 255–264. DOI: 10.1145/1374376.1374415.

[28] T. Sauerwald and A. Stauffer. “Rumor spreading and vertex expansion on regular graphs”. In: *Proceedings of the Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms*. SIAM, Philadelphia, PA, 2011, pp. 462–475.

[29] N. C. Wormald. “Models of random regular graphs”. In: *Surveys in combinatorics, 1999 (Canterbury)*. Vol. 267. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1999, pp. 239–298.

[30] N. C. Wormald. “The differential equation method for random graph processes and greedy algorithms”. English. In: *Lectures on Approximation and Randomized Algorithms*. PWN, Warsaw, 1999, pp. 73–155.

Sidon-Ramsey & B_h-Ramsey numbers

Manuel A. Espinosa-García[1*], Amanda Montejano[2†], Edgardo Roldán-Pensado[3*], J. David Suárez[4†]

[*]Centro de Ciencias Matemáticas, UNAM Campus Morelia, Morelia, Mexico
[†]Facultad de Ciencias, UNAM campus Juriquilla, Queretaro, Mexico

This research is supported by CONACyT project 282280 and PAPIIT project IG100822.

Abstract

Given a positive integer k, the Sidon-Ramsey number $\mathrm{SR}(k)$ is defined as the minimum n such that in every partition of $[1,n]$ into k parts there is a part containing two pairs of numbers with the same sum. In this note we recall what is known about these numbers and study two of their generalizations.

1 Introduction

A subset S of an additive group G is called a *Sidon set* if the sums of any two elements (possibly equal) of S are distinct. In other words, if $x,y,z,w \in S$ satisfy

$$x+y=z+w,$$

then $\{x,y\}=\{z,w\}$, which means that the equation above has only trivial solutions in S. For a given $X \subset G$, an important problem is to determine the maximum size of a Sidon set contained in X. This problem has been mainly studied when $G=\mathbb{Z}$ and $X=[1,n]$. We use $\mathrm{F}_2(n)$ to denote the size of the largest Sidon set contained in $[1,n]$. It is known that

$$n^{1/2}(1-o(1)) \leq \mathrm{F}_2(n) \leq n^{1/2}+0.998\, n^{1/4}. \tag{1}$$

The upper bound has been progressively improved [9, 14, 7], the best being recently established by Balogh, Füredi and Roy [2]. The lower bound may be inferred from several known constructions; in particular the one provided by Singer concerning maximal Sidon sets in $X=G=\mathbb{Z}_n$, where $n=q^2+q+1$ and q is a prime power [20]. For more information about problems related with Sidon sets the reader may consult the survey paper of O'Bryant [16].

As with many density theorems, there is a Ramsey version of the problem of maximizing the size of a Sidon set. For a given positive integer k, a *Sidon k-partition* of $X \subset G$ is a partition of X into k parts, all of which are Sidon sets. Let $\mathrm{SR}(k)$ be the *Sidon-Ramsey number*, defined as the minimum n such that there is no Sidon k-partition of $[1,n]$. This parameter can be found in different contexts under different names. For instance, the existence of the Sidon-Ramsey numbers is a consequence of a theorem of Rado [18] with the matrix

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 \end{pmatrix},$$

[1]Email: esgama93@gmail.com.
[2]Email: amandamontejano@ciencias.unam.mx.
[3]Email: e.roldan@im.unam.mx.
[4]Email: suar_david@hotmail.com.

where two new variables and two new equations are introduced to assure that no trivial solutions to the equation $x + y = z + w$ are considered. Liang et al. [13] and Xu et al. [21], using computer assistance, found the exact values of $\mathrm{SR}(k)$ for $k \leq 5$ and gave specific bounds for $k \leq 19$. They use the fact that $\mathrm{SR}(k)$ can be bounded from above using the pigeonhole principle, that is,

$$\mathrm{SR}(k) \leq (t-1)k + 1 \text{ for any } k, t \geq 2 \text{ satisfying } (\mathrm{F}_2(t) - 1)/(t-1) > k. \tag{2}$$

In the following Theorem, we compute the upper bound implied by (2) and (1), and also use the particular properties of the Sidon set constructed by Singer [20] to produce a lower bound for the Sidon-Ramsey numbers. This is done using ideas similar to the ones used by Chung and Graham [5]. Its proof is almost identical to the proof of Theorem 2 below.

Theorem 1. *Let k be a positive integer, then*

$$k^2 - O(k^c) \leq \mathrm{SR}(k) \leq k^2 + Ck^{3/2} + O(k),$$

where $c \leq 1.525$ depends on the distribution of the prime numbers and C can be taken close to 1.996 and depends on the best upper bound for Sidon numbers.

Therefore, the Sidon-Ramsey numbers $\mathrm{SR}(k)$ behave asymptotically as k^2.

In this note we present a couple of generalizations of Theorem 1. In the first, we consider the Ramsey analogue of a generalization of Sidon numbers called B_h sets. In the second, we consider the problem in higher dimensions.

2 B_h-Ramsey numbers in $\mathbb{N}$

A subset S of an additive group G is called a *B_h-set* if all sums of the form $a_1 + \cdots + a_h$, where $a_1, \ldots, a_h \in A$, are distinct; note that a B_2-set is a Sidon set. We use $\mathrm{F_h}(n)$ to denote the size of the largest B_h-set contained in $[1, n]$. It is known that

$$(1 + o(1))n^{1/h} \leq \mathrm{F_h}(n) \leq c_h n^{1/h}. \tag{3}$$

The lower bound was proved by Bose and Chowla [3], while the constant c_h in the upper bound has been successively improved [8, 12, 4, 6]. Currently, the best bounds are due to Green [11], who proved that $c_3 < 1.519$, $c_4 < 1.627$ and $c_h \leq 1/2e\,(h + (3/2 + o(1))\,ln(h))$.

As with Sidon sets, we can define the Ramsey version of this problem. For a given positive integer k and a subset X of an additive group G, a *B_h k-partition* of X is a partition of X into k parts, all of which are B_h-sets. Let $\mathrm{BR}_h(k)$ be the *B_h-Ramsey number*, defined as the minimum n such that there is no B_h k-partition of $[1, n]$.

The following theorem gives bounds for these numbers. The lower bound comes from considering translates of the B_h-set constructed by Ruzsa (for $h = 2$) [19] and by Gómez-Trujillo (for $h > 2$) [10]. The upper bound can be derived from Green's bound [11] and the pigeonhole principle.

Theorem 2. *Let k be a positive integer, then*

$$k^{\frac{h}{h-1}} - O\left(k^{1+\frac{c}{h-1}}\right) \leq \mathrm{BR}_h(k) \leq C_h k^{\frac{h}{h-1}},$$

where $c \leq 0.525$ depends on the distribution of the prime numbers and C_h depends on the best upper bound for $\mathrm{F_h}$ numbers.

Proof. To get the lower bound we use the construction on B_2-sets from Ruzsa [19] and B_h-sets (for $h > 2$) from Gómez and Trujillo [10]. Such constructions are given in $\mathbb{Z}_{p^h - p}$ where p is a prime number. Taking some translations of such sets we get a B_2 p-partition of Z_{p^2-p} and B_h $p^{h-1} - 1$-partition of

$\mathbb{Z}_{p^h-p}$ (for $h>2$). Then we have partitions for the intervals $[1,p^h-p]$. To get a result for intervals in general, we use the fact that there is a prime number in the interval $[x-x^{0.525},x]$ (see [1]).

We get an upper bound doing the computing given by the pigeonhole principle and the bounds in equation (3). □

Therefore, the B_h-Ramsey number $\mathrm{BR}_h(n)$ behaves asymptotically as $k^{h/h-1}$.

3 Higher dimensions

Another important task is to study B_h-sets in higher dimensions. For a fixed positive integer d and integers $n_1 \leq \cdots \leq n_d$, we denote by $\mathrm{F_h}(n_1,\ldots,n_d)$ the largest cardinality of a B_h-set in the d-dimensional box $X=\prod_{j=1}^{d}[1,n_j]\subset\mathbb{Z}^d$.

There is a natural way to map one-dimensional B_h-sets to d-dimensional B_h-sets, which implies

$$\mathrm{F_h}(n_1\cdots n_d)\leq \mathrm{F_h}(n_1,\ldots,n_d),$$

so the lower bounds in the d-dimensional problems come from the lower bounds of the one-dimensional problems.

When $n_1=\cdots=n_d=n$ and $h=2$, i.e. Sidon sets in high dimensions, Lindström [15] proved that

$$\mathrm{F}_2(n,\ldots,n)\leq n^{d/2}\left(1+o(1)\right).$$

When $h>2$, Rackham and Šarka [17] gave the best upper bound so far:

$$\mathrm{F_h}(n,\ldots,n)\leq \begin{cases} n^{\frac{d}{h}}t^{\frac{d}{h}}(t!)^{1/t}+O\left(n^{\frac{d^2}{h(d+1)}}\right) & \text{if } h=2t,\\ n^{\frac{d}{h}}t^{\frac{d-1}{h}}(t!)^{\frac{2}{h}}+O\left(n^{\frac{d^2}{h(d+1)}}\right) & \text{if } h=2t-1. \end{cases}$$

The best upper bound for the general case is due to Cilleruelo [7], he proved that

$$\mathrm{F}_2(n_1,\ldots,n_d)\leq N^{1/2}\left(1+O\left(\left(\frac{N_{s-1}}{N^{1/2}}\right)^{\frac{1}{d-s+2}}\right)\right),$$

where $N_0=1$, $N_i=\prod_{j=1}^{i}n_j$ for $1\leq i\leq d$, $N=N_d$ and s is the least integer such that $N^{1/2}\leq n_s^{d-s+2}N_{s-1}$.

In order to provide lower bounds for the B_h-Ramsey numbers in boxes, we provide upper bounds for $\mathrm{F_h}(n_1,\ldots,n_d)$. For this we use two lemmas, the first is about additive energy and appears in [7]. Let $r_X(g)=|\{(x,y):x,y\in X, x-y=g\}|$.

Lemma 3. *Let G be an additive group and let $A,B\subset G$. Then*

$$|A|^2\leq\frac{|A+B|}{|B|^2}\sum_{g\in G}r_A(g)r_B(g).$$

The second lemma is a generalization of Lemmas 4.2.1 and 4.3.1 in [17].

Lemma 4. *Let $B=[0,i_1-1]\times[0,i_2-1]\times\cdots\times[0,i_d-1]$.*

1. *If $h=2t$ then*

$$\sum_{z\in\mathbb{Z}^d}r_{kA}(z)r_B(z)\leq|B|^2+O(|B||A|^{h-1}).$$

2. If $h = 2k-1$ then

$$\sum_{z\in\mathbb{Z}^d} r_{k*A}(z)r_B(z) \le \frac{|A|}{k}|B|^2 + O(|B||A|^h).$$

Using these lemmas together with techniques previously used by Cilleruelo [7] and Rackham and Šarka [17] we obtain the following theorem.

Theorem 5. *Let $n_1 \le n_2 \le \cdots \le n_d$ be positive integers, set $N_0 = 1$, $N_i = \prod_{j=1}^{i} n_j$ for $1 \le i \le d$, $N = N_d$, and let s be the least integer such that $N^{1/h} \le n_s^{d-s+2} N_{s-1}$. Then, for $h \ge 2$,*

$$\mathrm{F_h}(n_1,\dots,n_d) \le \begin{cases} (t!)^{\frac{2}{h}} t^{\frac{d}{h}} N^{\frac{1}{h}} \left(1 + O\left(\left(\frac{N_{s-1}}{N^{\frac{1}{h}}}\right)^{\frac{1}{d-s+2}}\right)\right) & \text{if } h = 2t, \\ (t!)^{\frac{2}{h}} t^{\frac{d-1}{h}} N^{\frac{1}{h}} \left(1 + O\left(\left(\frac{N_{s-1}}{N^{\frac{1}{h}}}\right)^{\frac{1}{d-s+2}}\right)\right) & \text{if } h = 2t-1. \end{cases}$$

For the Ramsey version it is convenient to bound the size of the partition in terms of the dimensions of the box. Let $\mathrm{br}_h(n_1,\dots,n_d)$ be the largest positive integer k such that there is no B_h k-partition of $\prod_{i\le d}[1,n_i]$. In dimension $d=1$, $\mathrm{br}_h(n)$ is the counterpart of $\mathrm{BR}_h(k)$ in the sense that $\mathrm{br}_h(\mathrm{BR}_h(k)) = k$.

The second result of this section is a lower and upper bound for $\mathrm{br}_h(n_1,\dots,n_d)$.

Theorem 6. *Let $n_1 \le n_2 \le \cdots \le n_d$ be positive integers. Let $N_0 = 1$, $N_i = \prod_{j\le i} n_j$ for $1 \le i \le d$, $N = N_d$, and s the least index such that $N^{1/h} \le n_s^{d-s+2} N_{s-1}$. Then*

$$N^{\frac{h-1}{h}}\left(1 + O\left(N\right)^{\frac{-1+c}{h}}\right) \ge \mathrm{br}_h(n_1,\dots,n_d) \ge \begin{cases} \frac{N^{\frac{h-1}{h}}}{(k!)^{\frac{2}{h}} k^{\frac{d}{h}}}\left(1 - O\left(\left(\frac{N_{s-1}}{N^{\frac{1}{h}}}\right)^{\frac{1}{d-s+2}}\right)\right) & \text{if } h = 2k, \\ \frac{N^{\frac{h-1}{h}}}{(k!)^{\frac{2}{h}} k^{\frac{d-1}{h}}}\left(1 - O\left(\left(\frac{N_{s-1}}{N^{\frac{1}{h}}}\right)^{\frac{1}{d-s+2}}\right)\right) & \text{if } h = 2k-1, \end{cases}$$

where $c \le 0.525$ depends on the distribution of the prime numbers.

Proof. The lower bound is obtained by using the pigeonhole principle with the bounds given in Theorem 5. The upper bound follows from the lower bound in Theorem 2 and the fact that any B_h k-partition in $[1,N]$ can be mapped to a B_h k-partition in $[1,n_1]\times\cdots\times[1,n_d]$. □

Therefore, the B_h-Ramsey number $\mathrm{br}_h(n_1,\dots,n_d)$ behaves asymptotically as $N^{(h-1)/h}$.

References

[1] R. C. Baker, G. Harman, and J. Pintz. "The difference between consecutive primes. II". In: *Proc. London Math. Soc. (3)* 83.3 (2001), pp. 532–562. DOI: 10.1112/plms/83.3.532.

[2] J. Balogh, Z. Füredi, and S. Roy. *An upper bound on the size of Sidon sets*. Preprint. 2021. DOI: 10.48550/ARXIV.2103.15850.

[3] R. C. Bose and S. Chowla. "Theorems in the additive theory of numbers". In: *Comment. Math. Helv.* 37 (1962), pp. 141–147. DOI: 10.1007/BF02566968.

[4] S. Chen. "On the size of finite Sidon sequences". In: *Proc. Amer. Math. Soc.* 121.2 (1994), pp. 353–356. DOI: 10.2307/2160407.

[5] F. R. K. Chung and R. L. Graham. "On multicolor Ramsey numbers for complete bipartite graphs". In: *J. Combinatorial Theory Ser. B* 18 (1975), pp. 164–169. DOI: 10.1016/0095-8956(75)90043-x.

[6] J. Cilleruelo. "New upper bounds for finite B_h sequences". In: *Adv. Math.* 159.1 (2001), pp. 1–17. DOI: 10.1006/aima.2000.1961.

[7] J. Cilleruelo. "Sidon sets in $\mathbb{N}^d$". In: *J. Combin. Theory Ser. A* 117.7 (2010), pp. 857–871. DOI: 10.1016/j.jcta.2009.12.003.

[8] A. G. D'yachkov and V. V. Rykov. "B_s-sequences". In: *Mat. Zametki* 36.4 (1984), pp. 593–601.

[9] P. Erdös and P. Turán. "On a problem of Sidon in additive number theory, and on some related problems". In: *J. London Math. Soc.* 16 (1941), pp. 212–215. DOI: 10.1112/jlms/s1-16.4.212.

[10] C. A. Gómez Ruiz and C. A. Trujillo Solarte. "A new construction of modular B_h-sequences". In: *Mat. Enseñ. Univ. (N. S.)* 19.1 (2011), pp. 53–62.

[11] B. Green. "The number of squares and $B_h[g]$ sets". In: *Acta Arith.* 100.4 (2001), pp. 365–390. DOI: 10.4064/aa100-4-6.

[12] X. D. Jia. "On finite Sidon sequences". In: *J. Number Theory* 44.1 (1993), pp. 84–92. DOI: 10.1006/jnth.1993.1037.

[13] M. Liang, X. Li, B. Xiu, and X. Xu. "On Sidon-Ramsey numbers". In: *Journal of Computational and Theoretical Nanoscience* 10 (2013), pp. 884–887. DOI: 10.1166/jctn.2013.2784.

[14] B. Lindström. "An inequality for B_2-sequences". In: *J. Combinatorial Theory* 6 (1969), pp. 211–212.

[15] B. Lindström. "On B_2-sequences of vectors". In: *J. Number Theory* 4 (1972), pp. 261–265. DOI: 10.1016/0022-314X(72)90052-2.

[16] K. O'Bryant. "A complete annotated bibliography of work related to Sidon sequences". In: *Electron. J. Combin.* DS11.Dynamic Surveys (2004), p. 39.

[17] L. Rackham and P. Šarka. "B_h sequences in higher dimensions". In: *Electron. J. Combin.* 17.1 (2010), Research Paper 35, 15.

[18] R. Rado. "Note on combinatorial analysis". In: *Proc. London Math. Soc. (2)* 48 (1943), pp. 122–160. DOI: 10.1112/plms/s2-48.1.122.

[19] I. Z. Ruzsa. "Solving a linear equation in a set of integers. I". In: *Acta Arith.* 65.3 (1993), pp. 259–282. DOI: 10.4064/aa-65-3-259-282.

[20] J. Singer. "A theorem in finite projective geometry and some applications to number theory". In: *Trans. Amer. Math. Soc.* 43.3 (1938), pp. 377–385. DOI: 10.2307/1990067.

[21] X. Xu, M. Liang, and H. Luo. *Ramsey theory*. Unsolved problems and results. De Gruyter, Berlin; University of Science and Technology of China Press, [Hefei], 2018, pp. xii+178. DOI: 10.1515/9783110576702.

Powers of Hamilton cycles in graphs perturbed by a random geometric graph

Alberto Espuny Díaz[1*], Joseph Hyde[2†]

[*]Institut für Mathematik, Technische Universität Ilmenau, 98684 Ilmenau, Germany
[†]Mathematics Institute, Zeeman Building, University of Warwick, Coventry CV4 7AL, UK

Abstract

A random geometric graph $G^d(n,r)$ is obtained by placing n vertices uniformly and independently at random in the hypercube $[0,1]^d$ and joining two vertices by an edge if the distance between them is at most r. We study the problem of the containment of k-th powers of Hamilton cycles in the union of $G^d(n,r)$ with an n-vertex graph H_n with minimum degree αn. For all values of k, d and α, we provide asymptotically optimal values for r which ensure the union contains the k-th power of a Hamilton cycle with high probability. Our result implies asymptotically optimal conditions for the containment of other spanning structures.

1 Introduction

The theory of randomly perturbed graphs has been quickly developing in the last few years as a way to interpolate between extremal results and results about random graphs. For instance, a classical theorem of Dirac asserts that every graph H on $n \geq 3$ vertices with minimum degree $\delta(H) \geq n/2$ contains a Hamilton cycle (this minimum degree condition cannot be improved), and Pósa proved that there exists some contant C such that, if $p \geq C \log n/n$, then a.a.s. (that is, with probability tending to 1 as $n \to \infty$) $G(n,p)$ contains a Hamilton cycle (and this is false if $p \leq (1-\varepsilon)\log n/n$). Interpolating between these results, Bohman, Frieze and Martin [5] showed that, for any $\alpha \in (0,1/2)$, there exists some constant $C = C(\alpha)$ such that, if $p \geq C/n$, then, for any n-vertex graph H_n with minimum degree $\delta(H_n) \geq \alpha n$, a.a.s. $H_n \cup G(n,p)$ contains a Hamilton cycle. Note that the required p is improved by a logarithmic factor with respect to what is required in the purely random setting.

Since the seminal result of Bohman, Frieze and Martin [5], a lot of research has been devoted to understanding the behaviour of randomly perturbed graphs, especially with respect to the containment of different spanning structures (see, e.g., [1, 2, 4, 6, 7, 8, 9, 10, 12, 15, 16, 19, 20, 22]). All of these papers consider graphs H perturbed by a binomial random graph $G(n,p)$ or its Erdős-Rényi counterpart $G(n,m)$. Only very recently some authors started considering perturbations coming from other random graph models, such as random regular graphs or random geometric graphs. In this paper, it is the latter model that we are interested in.

For $n, d \in \mathbb{N}$ and $r > 0$, a random geometric graph $G^d(n,r)$ is a graph on vertex set $[n] \coloneqq \{1,\dots,n\}$ obtained as follows. Let $X_1,\dots,X_n$ be independent uniform random variables on $[0,1]^d$. Then, we take $\{\{i,j\} \in [n]^{(2)} : \|X_i - X_j\| \leq r\}$ to be the edge set of the random graph, where $\|\cdot\|$ denotes the Euclidean norm. (We remark that the same graph model may be defined for other norms; all our results

[1]Email: alberto.espuny-diaz@tu-ilmenau.de Research supported by the Carl Zeiss Foundation and by DFG grant PE 2299/3-1.

[2]Email: joseph.hyde@warwick.ac.uk Research supported by the UK Research and Innovation Future Leaders Fellowship MR/S016325/1.

extend to all ℓ_p norms with $1 \leq p \leq \infty$.) Díaz, Mitsche and Pérez-Giménez [11] showed that there exists a constant C such that, if $r \geq C(\log n/n)^{1/2}$, then a.a.s. $G^2(n,r)$ contains a Hamilton cycle (in fact, they determined the best possible C). Their result was strengthened and extended to all fixed $d \geq 2$ independently by Balogh, Bollobás, Krivelevich, Müller and Walters [3] and by Müller, Pérez-Giménez and Wormald [21], proving that $r \geq C(\log n/n)^{1/d}$ suffices for $G^d(n,r)$ to a.a.s. contain a Hamilton cycle (where C depends on d). The same result also holds when $d = 1$. Espuny Díaz [13] later provided a result analogous to that of Bohman, Frieze and Martin [5] when the random perturbation is given by a random geometric graph. Given an n-vertex graph H_n, we label its vertices with $[n]$ and consider the randomly perturbed graph $H_n \cup G^d(n,r)$ with edge set $E(H_n) \cup \{\{i,j\} \in [n]^{(2)} : \|X_i - X_j\| \leq r\}$. Espuny Díaz [13] showed that, for any $\alpha \in (0,1/2)$ and $d \geq 1$, there exists some contant $C = C(\alpha, d)$ such that, if $r \geq Cn^{-1/d}$, then for any graph H_n with minimum degree $\delta(H_n) \geq \alpha n$ a.a.s. $H_n \cup G^d(n,r)$ contains a Hamilton cycle. Similarly to the result of Bohman, Frieze and Martin, this improves upon the threshold in the purely random setting by a logarithmic factor.

In this paper, we present further results about graphs perturbed by a random geometric graph. In particular, we consider powers of Hamilton cycles (the k-th power of a cycle is obtained by adding an edge between any two vertices whose graph-distance in the cycle is at most k). Powers of Hamilton cycles have received much attention in the literature. In the extremal setting, Komlós, Sárközy and Szemerédi [17, 18] proved that, for n sufficiently large, any n-vertex graph H with minimum degree $\delta(H) \geq kn/(k+1)$ contains the k-th power of a Hamilton cycle. In the purely random setting, for all $d, k \geq 1$ there exists some constant $C = C(d,k)$ such that, if $r \geq C(\log n/n)^{1/d}$, then a.a.s. $G^d(n,r)$ contains the k-th power of a Hamilton cycle (this follows trivially from the results about Hamiltonicity and the triangle inequality: indeed, if $G^d(n,r)$ contains a Hamilton cycle, then $G^d(n,kr)$ must contain its k-th power). Our main result interpolates between these two.

Theorem 1. *For any $\alpha \in (0,1)$ and any positive integers d and k, there exists a constant C such that the following holds. For any n-vertex graph H_n with minimum degree $\delta(H_n) \geq \alpha n$, if $r \geq Cn^{-1/d}$, a.a.s. $H_n \cup G^d(n,r)$ contains the k-th power of a Hamilton cycle.*

For each positive k, this result is best possible for all $\alpha \in (0, k/(k+1))$. Indeed, one may consider the complete $(k+1)$-partite graph H_n were one part has size $\max\{0, (\alpha-1)k+1\}n$ and all other parts have the same size. A moment of thought reveals that this graph cannot contain the k-th power of a Hamilton cycle unless linearly many edges are added to it. And it is not hard to show that, if $r = o(n^{-1/d})$, then $G^d(n,r)$ will not contain sufficiently many edges for this to hold.

It is worth comparing Theorem 1 with the results about powers of Hamilton cycles in graphs perturbed by $G(n,p)$. There have been several papers exploring the extent to which powers of Hamilton cycles appear in randomly perturbed graphs for different ranges of α and p [1, 2, 7, 10, 12], but so far we do not have a full answer as to what the best possible asymptotic value of p is for each k and α. The full picture is only known for $k = 2$ [10], and it turns out that the necessary p presents 'jumps' as a function of α (and, in fact, it presents infinitely many such 'jumps'). One should expect a similar behaviour for higher values of k. In contrast to this, our result shows that random geometric graphs present a simpler behaviour, and completely solves the problem in this setting.

2 Ideas of the proof of Theorem 1

Due to the space limitations, we cannot show all the details of the proof. We thus explain the main ideas, and hope that they can paint a picture of the proof. The details can be found in the full version of this paper [14]. Let us begin by describing a direct proof of the fact that, if $r \geq C(\log n/n)^{1/d}$ for some sufficiently large $C = C(k,d)$, then a.a.s. $G^d(n,r)$ will contain the k-th power of a Hamilton cycle.

Fix some $k \geq 1$ and $d \geq 1$, let $r = C(\log n/n)^{1/d}$ (where C will be chosen later), and consider the hypercube $[0,1]^d$. We partition this hypercube into smaller hypercubes of side s, where we take $s = 1/m$ for some integer $2d/r \leq m \leq 2d/r + 1$. We refer to the resulting subhypercubes as *cells*. Our choice

of s guarantees that $[0,1]^d$ is indeed partitioned into hypercubes and that, for any two resulting cells c_1 and c_2 which share at least one point in their boundaries (in which case we say that c_1 and c_2 are *friends*), any two points $x, y \in c_1 \cup c_2$ satisfy that $\|x - y\| \leq r$.

Consider the random variables $X_1, \ldots, X_n$ which determine our random geometric graph. For each of the cells, the number of vertices whose position is assigned to some point inside the cell is given by a binomial random variable $\mathrm{Bin}(n, s^d)$. (Note that the probability that any point lies in the boundary of a cell is 0, hence we will assume that this is not the case throughout.) By using standard concentration inequalities and a union bound, we may show that, if C is taken to be sufficiently large, a.a.s. every cell will contain at least $2k$ vertices. Condition on this event. From now on, the proof is purely deterministic.

Consider an auxiliary graph Γ whose vertex set is the set of cells. Two cells are joined by an edge in Γ if they are friends. By definition, Γ has a grid-like structure. In particular, we may find a spanning path in Γ. We use this path to construct the desired k-th power of a Hamilton cycle. Indeed, use the spanning path to label the cells of Γ as $c_1, \ldots, c_{s^{-d}}$. We define a Hamilton cycle on the vertices of the random graph as follows. Let P_0 be empty. For each $i \in [s^{-d}]$, choose an arbitrary set of k vertices inside c_i and add them to P_{i-1}, in an arbitrary order, to create a longer path P_i. Recall that this is indeed a path by the definition of the cells. By the end of this step, we have a path of length ks^{-d}. Now, for each $i \in [s^{-d}]$, add all the remaining vertices in $c_{s^{-d}-i+1}$ to $P_{s^{-d}+i-1}$, in an arbitrary order, to create a longer path $P_{s^{-d}+i}$. Recall that, by the event we conditioned upon, we are guaranteed that at least k vertices are added to the path in each step. At the end of this process, the resulting path $P_{2s^{-d}}$ contains all n vertices, and we may close it into a Hamilton cycle $\mathcal{C}$. Furthermore, since at least k vertices were added to the path in every one of the steps above, we are guaranteed that, for every vertex v in the cycle, the following k vertices in $\mathcal{C}$ are either in the same cell as v or in one of its friends. This guarantees that $G^d(n, r)$ actually contains the k-th power of $\mathcal{C}$.

While we have chosen a spanning path in Γ to construct our Hamilton cycle $\mathcal{C}$, we remark that, in fact, any spanning tree of Γ may be used. Given any tree, the labelling of the cells can be obtained via a depth-first search, and then a modified version of the algorithm we have described can be used to ensure that all cells are traversed and obtain a Hamilton cycle $\mathcal{C}$. In this setting, cells may be visited more than twice, though, so we need to ensure that every cell contains sufficiently many vertices. The needed number of vertices is at most $2\Delta k$, where Δ is the maximum degree of the chosen tree, so it suffices to guarantee that every cell contains at least that many. By adjusting the value of C, we can guarantee that this will be true in all cells; the fact that the maximum degree of Γ is bounded by a function of d allows this to work for any given subtree of Γ.

Let us now consider the setting of randomly perturbed graphs. We want to largely mimic the ideas of the previous proof. In particular, once we fix $r = Cn^{-1/d}$ (for some constant C to be defined), we split $[0,1]^d$ into cells of side s, with s defined as above. If we could prove in this setting that every cell will contain at least $2k$ vertices, we would be able to proceed exactly as above. This, however, is not the case: a.a.s. there will be some cells which contain fewer than $2k$ vertices, and even some empty cells. Thus, we split our cells into two types: those that contain at least R vertices, called *dense* cells, and those that contain fewer, called *sparse* cells. Here, R is a sufficiently large constant that will guarantee that our process works. By adjusting the value of C and applying martingale concentration inequalities, we can show that a.a.s. the number of sparse cells is at most βn, where $\beta > 0$ is a constant that we can make as small as we want by adjusting C.

Consider now an auxiliary graph Γ. It is defined in a similar way as it was in the denser setting above, with the difference that its vertex set is restricted exclusively to the set of *dense* cells. The resulting graph need not be connected, but we can show that it will not have too many connected components: upon conditioning on the number of sparse cells being bounded as above, the number of components of Γ can be bounded by γn, where again $\gamma > 0$ can be taken arbitrarily small by adjusting the value of C. We now may take an arbitrary spanning tree of each of the connected components and, by the choice of R, use the algorithm above to construct the k-th power of a cycle containing all the vertices that lie in this component. Overall, this allows us to incorporate most of the vertices into k-th powers of cycles.

This is, however, far from our goal: not all vertices lie in the structure that we have constructed, and we would like to have a unique Hamilton cycle whose k-th power is contained in the graph. In order to find our desired spanning structure, we will need to use the edges of the fixed graph H_n with $\delta(H_n) \geq \alpha n$. We need to achieve two things: first, we need to incorporate all the remaining vertices into our structure; second, we need to graft together the different powers of cycles which we have obtained throughout.

Let us focus first on the remaining vertices. The idea to deal with these is quite simple. Assume that v lies in a sparse cell (all vertices in dense cells are already part of some cycle). If we can find a dense cell which contains at least $2k$ H_n-neighbours of v, then, when constructing the cycle containing these vertices, we can make it so that these $2k$ vertices appear consecutively (recall that, when constructing the spanning cycle in a component of dense cells, we have a lot of freedom; in particular, recall that the vertices which were chosen and their order could be arbitrary, so we are simply now restricting the choices to suit us). Once this is achieved, we can simply 'insert' v into the spanning structure. After this is done for every vertex in a sparse cell, we have a spanning structure consisting of a union of k-th powers of cycles.

Of course, if done greedily, we might run into some issues. For instance, we might choose the same set of $2k$ neighbours for two distinct vertices v_1 and v_2 in sparse cells; if this occurs, we might not be able to proceed, since there is no guarantee that v_1 and v_2 are joined by an edge in H_n. In order to avoid this problem, we will make sure that all the sets of vertices which we choose are disjoint; we achieve this by making sure that no dense cell is 'used' more than once. Indeed, for each vertex v, we say that a cell is v-dense if it contains at least $2k$ H_n-neighbours of v, and say that it is v-sparse otherwise. Using martingale inequalities and adjusting again the value of C, we can show that, for every v, the number of v-sparse cells is at most δn, where we can make $\delta > 0$ arbitrarily small. Since there are 'few' vertices in sparse cells, this will allow us to use a different dense cell to incorporate each vertex in a sparse cell into one of the powers of cycles.

Lastly, it remains to combine all the powers of cycles we have into a single spanning k-th power of a cycle. In order to achieve this, we follow an idea similar to what we did for vertices in sparse cells (though a bit more involved). Indeed, consider one of the at most γn components of Γ. All vertices which lie in the cells of this component are contained in one of the cycles we have constructed. Say that we want to combine this cycle with a different one. Suppose that we can find a cell c in the chosen component of Γ and a different dense cell c' outside this component satisfying the following: c and c' contain sets of $2k$ vertices such that H_n spans a complete bipartite graph between these two sets. If this is the case, we can adapt the construction of the k-th power of cycles in the components containing c and c' to guarantee that the respective sets of $2k$ vertices will appear consecutively in each cycle. Once the cycles are constructed this way, we can use the edges of the complete bipartite graph to join the two structures into a unique k-th power of a cycle, by using the edges of H_n to go from the first k vertices in c to the first k in c', then following the cycle in the component containing c' until we reach the last k vertices in c', and then using the edges of H_n to join to the last k vertices in c. If we could do this iteratively until only one structure remains, we would be done.

In order to show that we may indeed follow this process, we need to further classify our cells. For any dense cell c, we say that another cell c' is c-dense if we can find our desired complete bipartite graph between these two cells. Otherwise, we say c' is c-sparse. (Our definitions here are actually more technical, but we avoid the details for clarity.) Using martingales and some simple arguments about subgraphs in dense graphs, and adjusting the value of C once more, we can show that, for every dense cell c, the number of c-sparse cells is at most εn, where $\varepsilon > 0$ can be made arbitrarily small. Using this fact, we may now iteratively join the different structures that we have constructed so far. Indeed, we may always choose a cycle containing fewer than half of the cells to join to some other cycle; since there are very few c-sparse cells, we are guaranteed to find a c-dense cell outside the component we are considering. As earlier, in order to guarantee that our process will work, we must make sure that the sets of vertices where we use edges of H_n do not intersect (and we also want these to be disjoint from those we used while incorporating the vertices in sparse cells). But, since there are very few components and very few c-sparse cells for each c, this can be achieved.

3 Applications

As a direct application of Theorem 1, we can obtain asymptotically optimal bounds for randomly perturbed graphs to contain other spanning structures. This allows us to extend several other lines of research in randomly perturbed graphs into the setting where the perturbation is given by a random geometric graph. These results follow directly from the fact that many spanning structures of interest are contained in some power of a Hamilton cycle.

This is the case, for instance, for F-factors, for any fixed graph F. (A graph G contains an *F-factor* if it contains a union of vertex-disjoint copies of F covering all vertices of G.) Indeed, given any fixed graph F and a Hamilton cycle, there must exist some $k \in \mathbb{N}$ such that any $|V(F)|$ consecutive vertices of the Hamilton cycle span a copy of F in its k-th power. Therefore, the following result is an immediate consequence of Theorem 1.

Corollary 2. *For every fixed graph F, every $\alpha \in (0,1)$, and every positive integer d, there exists a constant C such that the following holds. For any n-vertex graph H_n with minimum degree $\delta(H_n) \geq \alpha n$ and n divisible by $|V(F)|$, if $r \geq Cn^{-1/d}$, a.a.s. $H_n \cup G^d(n,r)$ contains an F-factor.*

This result is asymptotically optimal for some (but possibly not the whole) range of α (see [14, Section 4]). This extends results of [4, 8, 9, 15] to graphs perturbed by random geometric graphs.

As a different application, we consider 2-universality, that is, the property of containing every n-vertex graph of maximum degree at most 2 as a subgraph. For graphs perturbed by $G(n,p)$, this problem has been considered in [10, 22]. Since the square of a Hamilton cycle is 2-universal, Theorem 1 immediately implies the following result.

Corollary 3. *For every $\alpha \in (0,1)$ and every positive integer d, there exists a constant C such that, if H_n is an n-vertex graph with minimum degree $\delta(H_n) \geq \alpha n$ and $r \geq Cn^{-1/d}$, then a.a.s. $H_n \cup G^d(n,r)$ is 2-universal.*

This result is asymptotically optimal for all $\alpha \in (0, 2/3)$ and completely resolves the problem of 2-universality in graphs perturbed by a random geometric graph.

In more generality, our main result may be applied to obtain any spanning structure which is contained in the k-th power of a Hamilton cycle, for some sufficiently large (but fixed) k. Moreover, contrary to other settings (such as in graphs perturbed by $G(n,p)$), the results obtained are asymptotically optimal for most such spanning structures, for some range of α.

References

[1] S. Antoniuk, A. Dudek, C. Reiher, A. Rucinski, and M. Schacht. "High powers of Hamiltonian cycles in randomly augmented graphs". In: *Journal of Graph Theory* 98 (2021), pp. 255–284.

[2] S. Antoniuk, A. Dudek, and A. Ruciński. *Powers of Hamiltonian cycles in randomly augmented Dirac graphs – the complete collection*. Preprint. 2022. DOI: 10.48550/ARXIV.2204.10738.

[3] J. Balogh, B. Bollobás, M. Krivelevich, T. Müller, and M. Walters. "Hamilton cycles in random geometric graphs". In: *Ann. Appl. Probab.* 21.3 (2011), pp. 1053–1072. DOI: 10.1214/10-AAP718.

[4] J. Balogh, A. Treglown, and A. Z. Wagner. "Tilings in randomly perturbed dense graphs". In: *Combin. Probab. Comput.* 28.2 (2019), pp. 159–176. DOI: 10.1017/S0963548318000366.

[5] T. Bohman, A. Frieze, and R. Martin. "How many random edges make a dense graph Hamiltonian?" In: *Random Structures Algorithms* 22.1 (2003), pp. 33–42. DOI: 10.1002/rsa.10070.

[6] J. Böttcher, J. Han, Y. Kohayakawa, R. Montgomery, O. Parczyk, and Y. Person. "Universality for bounded degree spanning trees in randomly perturbed graphs". In: *Random Structures Algorithms* 55.4 (2019), pp. 854–864. DOI: 10.1002/rsa.20850.

[7] J. Böttcher, R. Montgomery, O. Parczyk, and Y. Person. "Embedding spanning bounded degree graphs in randomly perturbed graphs". In: *Mathematika* 66.2 (2020), pp. 422–447. DOI: 10.1112/mtk.12005.

[8] J. Böttcher, O. Parczyk, A. Sgueglia, and J. Skokan. *Triangles in randomly perturbed graphs*. 2020. DOI: 10.48550/ARXIV.2011.07612.

[9] J. Böttcher, O. Parczyk, A. Sgueglia, and J. Skokan. "Cycle factors in randomly perturbed graphs". In: *Procedia Computer Science* 195 (2021). Proceedings of the XI Latin and American Algorithms, Graphs and Optimization Symposium., pp. 404–411. DOI: https://doi.org/10.1016/j.procs.2021.11.049.

[10] J. Böttcher, O. Parczyk, A. Sgueglia, and J. Skokan. *The square of a Hamilton cycle in randomly perturbed graphs*. 2022. DOI: 10.48550/ARXIV.2202.05215.

[11] J. Díaz, D. Mitsche, and X. Pérez. "Sharp threshold for Hamiltonicity of random geometric graphs". In: *SIAM J. Discrete Math.* 21.1 (2007), pp. 57–65. DOI: 10.1137/060665300.

[12] A. Dudek, C. Reiher, A. Ruciński, and M. Schacht. "Powers of Hamiltonian cycles in randomly augmented graphs". In: *Random Structures Algorithms* 56.1 (2020), pp. 122–141. DOI: 10.1002/rsa.20870.

[13] A. Espuny Díaz. *Hamiltonicity of graphs perturbed by a random geometric graph*. 2021. DOI: 10.48550/ARXIV.2102.02321.

[14] A. Espuny Díaz and J. Hyde. *Powers of Hamilton cycles in dense graphs perturbed by a random geometric graph*. Preprint. 2022. DOI: 10.48550/ARXIV.2205.08971.

[15] J. Han, P. Morris, and A. Treglown. "Tilings in randomly perturbed graphs: bridging the gap between Hajnal-Szemerédi and Johansson-Kahn-Vu". In: *Random Structures Algorithms* 58.3 (2021), pp. 480–516. DOI: 10.1002/rsa.20981.

[16] F. Joos and J. Kim. "Spanning trees in randomly perturbed graphs". In: *Random Structures Algorithms* 56.1 (2020), pp. 169–219. DOI: 10.1002/rsa.20886.

[17] J. Komlós, G. N. Sárközy, and E. Szemerédi. "On the square of a Hamiltonian cycle in dense graphs". In: *Proceedings of the Seventh International Conference on Random Structures and Algorithms (Atlanta, GA, 1995)*. Vol. 9. 1-2. 1996, pp. 193–211. DOI: 10.1002/(SICI)1098-2418(199608/09)9:1/2<193::AID-RSA12>3.0.CO;2-P.

[18] J. Komlós, G. N. Sárközy, and E. Szemerédi. "Proof of the Seymour conjecture for large graphs". In: *Ann. Comb.* 2.1 (1998), pp. 43–60. DOI: 10.1007/BF01626028.

[19] M. Krivelevich, M. Kwan, and B. Sudakov. "Cycles and matchings in randomly perturbed digraphs and hypergraphs". In: *Combin. Probab. Comput.* 25.6 (2016), pp. 909–927. DOI: 10.1017/S0963548316000079.

[20] M. Krivelevich, M. Kwan, and B. Sudakov. "Bounded-degree spanning trees in randomly perturbed graphs". In: *SIAM J. Discrete Math.* 31.1 (2017), pp. 155–171. DOI: 10.1137/15M1032910.

[21] T. Müller, X. Pérez-Giménez, and N. Wormald. "Disjoint Hamilton cycles in the random geometric graph". In: *J. Graph Theory* 68.4 (2011), pp. 299–322. DOI: 10.1002/jgt.20560.

[22] O. Parczyk. "2-universality in randomly perturbed graphs". In: *European J. Combin.* 87 (2020), pp. 103118, 11. DOI: 10.1016/j.ejc.2020.103118.

Maximum running times for graph bootstrap percolation processes

David Fabian[1*], Patrick Morris[2†], Tibor Szabó[3*]

[*]Institute of Mathematics, Freie Universität Berlin, Germany
[†]Departament de Matemàtiques (MAT), Universitat Politècnica de Catalunya (UPC), Barcelona, Spain.

Abstract

Given a fixed graph H and an n-vertex graph G the *H-bootstrap percolation process* of H on G is defined to be the sequence of graphs G_i, $i \geq 0$ which starts with $G_0 := G$ and in which G_{i+1} is obtained from G_i by adding every edge that completes a copy of H. This process is an example of a cellular automata and has been extensively studied since being introduced by Bollobás in 1968. Recently, Bollobás raised the question of determining the maximum running time of this process, over all choices of n-vertex graph G. Here, the running time of the process is number of steps t the process takes before stabilising, that is, when $G_t = G_{t+1}$. Recent papers of Bollobás–Przykucki–Riordan–Sahasrabudhe, Matzke and Balogh–Kronenberg–Pokrovskiy–Szabó have addressed the case when H is a clique, and determined the asymptotics of this maximum running time for all cliques apart from K_5. Here, we initiate the study of the maximum running time for other graphs H and provide a survey of our new results in this direction. We study several key examples, giving precise results for trees and cycles, and giving general results towards understanding how the maximum running time of the H-bootstrap percolation process depends on properties of H, in particular exploring the relationship between this graph parameter and the degree sequence of H. Many interesting questions remain and along the way, we indicate some directions for future research.

1 Introduction

In 1968, Bollobás [7] introduced the notion of weakly saturated graphs. For a graph G, let $n_H(G)$ denote the number of copies of H in G. Then Bollobás defined an n-vertex graph G to be *weakly H-saturated* if there exists some ordering $e_1, \dots, e_\ell \in E(K_n) \setminus E(G)$ of the *non*-edges of G such that for $i = 1, \dots, \ell$, one has that $n_H(G \cup \{e_1, \dots, e_i\}) > n_H(G \cup \{e_1, \dots, e_{i-1}\})$. In words, adding the non-edges to G according to some order gives new copies of H at every step. Bollobás [7] asked the extremal question of determining the *weak saturation number* of H, that is, the minimum number of edges of an n-vertex weakly H-saturated graph, which we denote here by $\mathrm{wsat}_H(n)$. In particular, he was interested in cliques and conjectured that $\mathrm{wsat}_{K_r}(n) = \binom{n}{2} - \binom{n-r+2}{2}$. This was proved independently by Alon [2], Frankl [10] and Kalai [12]. We remark that the extremal construction of a weakly saturated graph with $\mathrm{wsat}_{K_r}(n)$ edges is given by removing a clique with $n - r + 2$ vertices from K_n. There has since been a wealth of results exploring weak saturation numbers and variants.

[1]Email: davidjf@fu-berlin.de. Research supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) Graduiertenkolleg "Facets of Complexity" (GRK 2434).

[2]Email: pmorrismaths@gmail.com. Research supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) Walter Benjamin program - project number 504502205.

[3]Email: szabo@math.fu-berlin.de. Research supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - The Berlin Mathematics Research Center MATH+ (EXC-2046/1, project ID: 390685689).

As noted by Bollobás, an equivalent way to define weak saturation is via the following dynamic process. Fix some n-vertex graph G. Then the *H-bootstrap (percolation) process* $(G_i)_{i\geq 0}$ with *initial graph* G is defined as follows. Fix $G_0 = G$ and for $i \geq 1$, let G_i be defined on the same vertex set as G such that

$$E(G_i) := E(G_{i-1}) \cup \{e \in E(K_n) \setminus E(G_{i-1}) : n_H\left(G_{i-1} \cup \{e\}\right) > n_H(G_{i-1})\}.$$

Note that this process will *stabilise* at some point with $\tilde{G} = G_t = G_{t+1} = \cdots$ for some n-vertex $\tilde{G}$ and $t \in \mathbb{N}$. We call $\tilde{G}$ the *final graph* of the process. Note also that a graph G is weakly H-saturated if and only if the H-bootstrap process with initial graph G has final graph K_n, in which case we say the process *percolates*. Viewing weak saturation in this way links the notion to the study of *cellular automata*, a deep topic introduced by von Neumann (see [17]) following a suggestion of Ulam [16]. Indeed, the general setup of a cellular automata is to study the spread of a virus through a (hyper-)graph where some vertices are initially infected and the virus is passed on to other vertices at each time step according to some local homogeneous rule. By considering the hypergraph whose vertex set is $E(K_n)$ and whose edge set encodes copies of H, one can view the H-bootstrap process as a cellular automata. The literature on cellular automata is vast and the topic has been studied from many different perspectives as the concept can describe important processes occurring in physics, sociology and computer science (see for example [1]).

In recent years, this study has become prominent in extremal and probabilistic combinatorics with techniques from these areas being successfully applied to address key problems from other areas (see for example the very nice survey of Morris [14]) as well as new lines of research in combinatorics being motivated from this connection. In particular, inspired by analogous questions for similar automata studied in physics [9], Balogh, Bollobás and Morris [3] initiated the study of the H-bootstrap process (and coined this terminology) when the starting graph G is the random graph $G_{n,p}$ and asked for the threshold probability at which the process with initial graph $G_{n,p}$ percolates. This has become a central theme in the study of graph percolation.

1.1 The running time of bootstrap processes

As discussed above, most of the research on the graph bootstrap percolation process has focused on whether or not the process *percolates*. Adopting the cellular automata view of a virus spreading, this translates to asking whether or not the virus will reach the whole population, which is certainly a natural line of investigation. In this abstract we will rather be interested in *how long* the virus will spread for, a question which one could also imagine being important in applications. This perspective, however, has been considerably less explored until recently. We mention results of Benevides and Przykucki [6, 15] studying the running time of graph neighbourhood percolation, a cellular automata closely related to the H-bootstrap process, and work of Gunderson, Koch and Przykucki [11] studying how long the H-bootstrap process lasts when the initial graph is random.

Here, we define the *running time* of the H-bootstrap process $(G_i)_{i\geq 0}$ with initial graph G to be $\tau_H(G) := \min\{t \in \mathbb{N} : G_t = G_{t+1}\}$, the time at which the process stabilises. Recently, Bollobás posed the natural extremal question of determining the *maximum* running time of the H-bootstrap process.

Definition 1. For $n \in \mathbb{N}$, we define $M_H(n)$ to be

$$M_H(n) := \max_{|V(G)|=n} \tau_H(G),$$

the maximum running time of the H-bootstrap process over all choices of initial graph G with n vertices.

Before stating the known results in this direction, we make some initial observations. Firstly, note that to study maximum running time, we no longer care if the bootstrap process percolates or not. It could be that the initial graph G that maximises the running time of the process, gives a process which

has a final graph which is incomplete. It is also worth noting that the extremal graph for the weak saturation numbers discussed in the opening paragraph of this introduction are poor candidates for having a long running time. Indeed, although they maximise the number of missing edges, the process stabilises in just one step!

The initial focus of research into maximum running times has been the case when H is a clique. When $H = K_3$ and G is a path with n vertices, one can see that $\tau_H(G) = \lceil \log_2(n) \rceil$ as the distance between any pair of non-adjacent vertices halves at each step. Moreover, this happens for any pair of vertices in each connected component of *any* initial graph and as the n-vertex path maximises the diameter of an n-vertex graph, we have $\tau_H(G) \leq \lceil \log_2(n) \rceil$ for all n-vertex G and hence $M_H(n) = \lceil \log_2(n) \rceil$. For K_4, the maximum running time is much larger. The following was shown by Bollobás, Przykucki, Riordan and Sahasrabudhe [8] and, independently, by Matzke [13].

Theorem 2. *The maximum running time of the K_4-bootstrap process in $M_{K_4}(n) = n - 3$, for all $n \geq 3$.*

Bollobás, Przykucki, Riordan and Sahasrabudhe [8] realised that the running times could be even longer for K_r-processes as r grows and they gave constructions showing that $M_{K_r}(n) \geq n^{2-\lambda_r - o(1)}$ for $r \geq 5$, where λ_r is some explicit constant such that $\lambda_r \to 0$ as $r \to \infty$. However the same authors believed that there was a limit to how long the the K_r-bootstrap process could last and conjectured that for all $r \geq 5$, $M_{K_r}(n) = o(n^2)$. It turns out that this conjecture was in fact false. Indeed, Balogh, Kronenberg, Pokrovskiy and the third author [4] of this abstract proved the following.

Theorem 3. *For all $r \geq 6$, we have that $M_{K_r}(n) \geq \frac{n^2}{2500}$.*

In contrast to the construction of Bollobás et al., which was probabilistic in nature, the authors of [4] used an explicit construction to achieve Theorem 3. Interestingly, the construction could not be pushed to give quadratic time for K_5, but they could show that $M_{K_5}(n) \geq n^{2-o(1)}$. Intriguingly, this lower bound comes from a connection to arithmetic number theory and in fact they show that $M_{K_5}(n) \geq \Omega(nr_3(n))$, where $r_3(n)$ denotes the size of the largest set in $[n]$ which contains no 3-term arithmetic progressions (configurations of the form $a, a+b, a+2b$ for $a, b \in \mathbb{N} \setminus \{0\}$). A famous construction of Behrend [5] gives that $r_3(n) \geq n^{1-O(1/\sqrt{\log n})}$, whilst it is well-known that $r_3(n) = o(n)$, as was originally shown by Roth in 1953. Determining the asymptotics of $M_{K_5}(n)$, and in particular, if it can be quadratic or not, remains a very interesting open problem.

The remainder of this extended abstract will be concerned with exploring the behaviour of the function $M_H(n)$ for graphs H which are not complete. All the results stated are novel and due to the authors. We believe that there are many interesting directions to explore in this area. We aim to give an overview of our findings, highlight interesting connections with other lines of research, and pose questions for further research. Whilst some short proofs and proof sketches are given, full proofs are omitted and will appear in forthcoming papers.

2 Trees and Cycles

We begin by studying some sparse graphs H, addressing the case when H is a tree or a cycle. First, let us look at the simple example of H being a star.

Example 4. For $t \in \mathbb{N}$ and n sufficiently large, we have that $M_{K_{1,t-1}}(n) = t - 1$.

Proof. In the $K_{1,t-1}$-process $(G_i)_{i \geq 0}$ with some arbitrary initial graph G, we say a vertex is *universal* at time i if it is connected to all other vertices in G_i and we call it *central* at time i if it plays the role of the centre of a copy of a star that lies in G_i but not in G_{i-1}. Note that if a vertex is central at time i then it is universal at time j for all $j \geq i$ and hence cannot be central at some time $k > i$. Hence at every step i, if $G_i \neq G_{i-1}$, then there is some vertex v that is central (and hence universal) in time i that was not universal at time $i - 1$. Therefore, if $\tau_{K_{1,t-1}}(G) \geq t - 2$, then G_{t-2} contains at least $t - 2$ universal vertices and G_{t-1} must be a complete graph. This shows that $M_{K_{1,t-1}}(n) \leq t - 1$.

The lower bound is attained when G the disjoint union of the stars $K_{1,s}$, $1 \leq s \leq t-2$ and $n - \binom{t}{2} + 1$ isolated vertices. For this initial graph G, the process runs for $t-1$ steps before stabilising at the complete graph. At each time step, one vertex becomes univeral and the degree of all other vertices (which are not already universal or already adjacent to the central vertex) increases by 1. □

In our first main result we explore the behaviour of $M_T(n)$ for an arbitrary tree T by showing that the running time of the T-bootstrap process is constant. That is, the maximum running time is independent of the number of vertices of the initial graph G and can be bounded by a function which depends only on the tree T itself. Specifically, we show the following.

Theorem 5. *There exists a constant C such that for any t-vertex tree T and $n \in \mathbb{N}$, we have that $M_T(n) \leq Ct^2$.*

As in Example 4, our proof of Theorem 5 relies on vertices that become (almost-)universal. For example, the vertices that play the role of the neighbour of a leaf of T in some copy of T during the process will become almost universal, as in the next step they will be adjacent to all vertices not in the copy of T. Once enough almost universal vertices have been found, one can use them to easily find copies of T and prove that the process reaches the complete graph. There are many subtleties that arise when working through the details and a very careful analysis is necessary, in particular to achieve an upper bound with a reasonable dependence on the number of vertices t of the tree. We believe that the quadratic dependence given in Theorem 5 may not be optimal and pose the question of finding the longest running time over all choices of trees T. A lower bound is given by the linear running time for the star in Example 4.

Question 6. *For $t \in \mathbb{N}$ (and $n \in \mathbb{N}$ large) what is the maximum of $M_T(n)$ over all choices of tree T with t vertices?*

For cycles, we determine the precise maximum running time, giving the first infinite family of graphs H for which $M_H(n)$ is completely determined.

Theorem 7. *For $3 \leq k \in \mathbb{N}$ and n sufficiently large, we have that*

$$M_{C_k}(n) = \begin{cases} \lceil \log_{k-1}(n + k^2 - 4k + 2) \rceil & , \ k \text{ odd} ; \\ \lceil \log_{k-1}(2n + k^2 - 5k) \rceil & , \ k \text{ even} . \end{cases}$$

As with $K_3 = C_3$, the logarithmic running time for odd cycles C_k stems from the distance between two vertices being divided by $k-1$ at each step, and the n-vertex path provides a construction that maximises the running time. In order to obtain the precise running time, we need to understand fully the 'gaps'; the non-edges that last the longest in the process. This leads to an arithmetic problem and $k^2 - 4k + 2$ is in fact the largest natural number that cannot be expressed as an integral linear combination of $k-2$ and k with non-negative coefficients. This is what is known as the *Frobenius number* of $k-2$ and k, an important concept in the study of numerical semigroups. Although the results (and proofs) for the cases of odd and even cycles are similar, we see a different behaviour which arises due to the fact that even cycles are bipartite. Indeed, note that for a bipartite graph H (that is 2-edge connected) and an initial graph that is also bipartite, the H-bootstrap process will remain bipartite throughout. By choosing an initial graph that is *almost bipartite*, this parity issue can be manipulated to affect the running time and understanding the effect of this is the crucial challenge to determining the running time for the even cycles. Indeed, the extremal construction for even cycles is a triangle with a long path appended to one of its vertices (for certain $n \in \mathbb{N}$ this construction actually needs to be tweaked slightly but is very similar in flavour).

3 Graphs with vertices of small degree

In the previous section, we saw how the maximum running times of trees is at most constant (in n), whilst cycles give a logarithmic maximum running time. It turns out that these fast running times are rare and rely on the simple degree structure of the graphs considered. Indeed, our next result shows that low degree vertices are necessary in order to have a fast maximum running time.

Theorem 8. *Let H be a connected graph with minimum degree $\delta(H) \geq 2$ and maximum degree $\Delta(H) \geq 3$. Then $M_H(n) = \Omega(n)$.*

Our construction for Theorem 8 is similar to the construction used to show linear running time for K_4. The initial graph is given by a small clique (of size $v(H)-1$) appended to a graph G' with bounded maximum degree, and in fact bounded *bandwidth*. That is, there is some linear order σ on the vertices of G' such that all the neighbours of a vertex v are close to v according to σ. We also take G' to have large *girth* (the length of the smallest cycle in G'). For example in the case when $\delta(H) = 2$, G' is simply a long path. As the bootstrap process runs, the clique that we start with grows, swallowing up each of the vertices of G' along the ordering of its vertices. However, by appealing to the bounded bandwidth and large girth of G', we can bound how fast the clique grows, showing that it only reaches some constant number of further vertices with each step and the other vertices of G' will remain untouched.

Theorem 8 shows that small degrees are necessary for fast running time. However, our next result shows that they are not *sufficient* and we give a graph H with linear running time and small degrees.

Proposition 9. *There exists a graph H with $\delta(H) = 1$ and $\Delta(H) = 3$ such that $M_H(n) = \Omega(n)$.*

Given the use of (almost-)universal vertices in our proof of the running time of trees, Proposition 9 may seem quite surprising. Our construction of H is explicit and can be split into a vertex v and two subgraphs H_1 and H_2 such that H_1 is attached to v (the degree 1 vertex) and H_1 and H_2 are attached by a single edge. The key to the construction is that we choose H_1 to be non-bipartite whilst H_2 is bipartite (and 2-connected). By choosing an initial graph G with one copy of H_1 and the rest of the graph G being bipartite, we can force that the H-bootstrap process on G essentially mimics the H_2-process on $G \setminus H_1$. Due to the fact that the $G_i \setminus H_1$ will remain bipartite, we can ensure that no further copies of H_1 occur and in every copy of H in the process, the H_1 part of H is fixed. This guarantees that the (unique) almost universal vertex will not be used elsewhere.

4 Dense graphs

So far, we have focused on graphs H for which the running time is fast. We can also look at the other extreme. The next result shows that when H is very dense, we can find initial graphs for which the running time is asymptotically maximal. The construction of the initial graph here builds on the ideas of Balogh, Kronenberg, Pokrovskiy and the third author in proving Theorem 3

Theorem 10. *Let H be a k-vertex graph with at least 6 vertices such that $\delta(H) \geq 3k/4$. Then $M_H(n) = \Omega(n^2)$.*

Theorem 10 suggests the following extremal question.

Question 11. *For large $k \in \mathbb{N}$, what is smallest minimum degree condition on a k-vertex H that guarantees that $M_H(n)$ is quadratic in n?*

In order to understand whether a H-bootstrap process can have quadratic running time, we give the following general result, which links $M_H(n)$ to the Turán number of H, which we denote by $ex(n; H)$, the largest number of edges in a H-free n-vertex graph.

Proposition 12. *For any connected graph H, we have that $M_H(n) \leq 2ex(n; H)$.*

Proof. Set $h := |E(H)|$. Let G be a graph on n vertices with $\tau_H(G) = M_H(n)$, and let $(G_i)_{i\geq 0}$ be the H-bootstrap process with initial graph G. For $0 \leq j \leq \lfloor M_H(n)/2 \rfloor$ choose an edge e_j from $E(G_{2j})\setminus E(G_{2j-1})$. The graph with vertex set $V(G)$ and edge set $\{e_0, \ldots, e_{\lfloor M_H(n)/2 \rfloor}\}$ is H-free: Suppose it contained a copy H' of H with edges $e_{j_1}, \ldots, e_{j_h}$ where $j_1 < \ldots < j_h$. Then $H' - e_{j_h}$ is a copy of H^- in $G_{2j_{h-1}}$, so by definition of the H-process, e_{j_h} would be present in $G_{2j_{h-1}+1}$, which contradicts the choice of e_{j_h}. □

The well-known Erdős-Stone-Simonovits theorem from Turán theory, implies that for any *bipartite graph* H, one has that $ex(n; H) = o(n^2)$ and hence $M_H(n) = o(n^2)$ due to Proposition 12. In particular, this gives a lower bound of $k/2$ for Question 11 by considering a complete bipartite graph $H = K_{k/2,k/2}$.

References

[1] J. Adler and U. Lev. "Bootstrap percolation: visualizations and applications". In: *Brazilian Journal of Physics* 33 (2003), pp. 641–644.

[2] N. Alon. "An extremal problem for sets with applications to graph theory". In: *Journal of Combinatorial Theory, Series A* 40.1 (1985), pp. 82–89.

[3] J. Balogh, B. Bollobás, and R. Morris. "Graph bootstrap percolation". In: *Random Structures & Algorithms* 41.4 (2012), pp. 413–440.

[4] J. Balogh, G. Kronenberg, A. Pokrovskiy, and T. Szabó. *The maximum length of K_r-Bootstrap Percolation*. Proceedings of the American Mathematical Society. To appear.

[5] F. A. Behrend. "On sets of integers which contain no three terms in arithmetical progression". In: *Proceedings of the National Academy of Sciences of the United States of America* 32.12 (1946), p. 331.

[6] F. Benevides and M. Przykucki. "Maximum percolation time in two-dimensional bootstrap percolation". In: *SIAM Journal on Discrete Mathematics* 29.1 (2015), pp. 224–251.

[7] B. Bollobás. "Weakly k-saturated graphs". In: *Beiträge zur Graphentheorie (Kolloquium, Manebach, 1967)*. 1968, pp. 25–31.

[8] B. Bollobás, M. Przykucki, O. Riordan, and J. Sahasrabudhe. "On the maximum running time in graph bootstrap percolation". In: *Electronic Journal of Combinatorics* 24.2 (2017).

[9] J. Chalupa, P. L. Leath, and G. R. Reich. "Bootstrap percolation on a Bethe lattice". In: *Journal of Physics C: Solid State Physics* 12.1 (1979), p. L31.

[10] P. Frankl. "An extremal problem for two families of sets". In: *European Journal of Combinatorics* 3.2 (1982), pp. 125–127.

[11] K. Gunderson, S. Koch, and M. Przykucki. "The time of graph bootstrap percolation". In: *Random Structures & Algorithms* 51.1 (2017), pp. 143–168.

[12] G. Kalai. "Weakly saturated graphs are rigid". In: *Convexity and graph theory, Annals of Discrete Mathematics*. 1984, pp. 189–190.

[13] K. Matzke. "The saturation time of graph bootstrap percolation". In: *arXiv preprint* (2015). DOI: 10.48550/ARXIV.1510.06156.

[14] R. Morris. "Bootstrap percolation, and other automata". In: *European journal of combinatorics* 66 (2017), pp. 250–263.

[15] M. Przykucki. "Maximal percolation time in hypercubes under 2-bootstrap percolation". In: *The Electronic Journal of Combinatorics* 19.2 (2012), P41.

[16] S. Ulam. "Random processes and transformations". In: *Proceedings of the International Congress on Mathematics 1950*. 1952, pp. 264–275.

[17] J. von Neumann. “Theory of self-reproducing automata”. In: *University of Illinois Press, Urbana* (1966).

Homomorphisms between graphs embedded in surfaces (extended abstract)

Delia Garijo[1*], Andrew Goodall[2†], Lluís Vena[3‡]

[*]Dpto. de Matemática Aplicada, Universidad de Sevilla
[†]Computer Science Institute, Charles University, Prague
[‡]Dept. de Matemàtiques, Universitat Politècnica de Catalunya

Abstract

A graph homomorphism can be realized as a sequence of vertex identifications followed by suppression of multiple edges. We extend this notion to cellularly embedded graphs in closed surfaces (maps) in a way that preserves both the combinatorial structure of the graph and the topological structure of the surface. By restricting vertex identifications to those that can be performed while respecting the surface topology, we obtain a definition of map homomorphism that preserves genus and orientability. Such notions as the core of a graph and the homomorphism order are then extended to maps. We characterize map cores via a combinatorial formulation of some contractible curves on the surface in which the graph is embedded. We also show that, in contrast to graph homomorphisms, the poset of map cores ordered by the existence of a homomorphism does not contain any dense interval (so it is not universal for countable posets), and give examples of a pair of cores with an infinite number of cores between them and of an infinite chain of gaps.

1 Introduction

Maps can be defined topologically or combinatorially. Perhaps the most familiar definition is that of a graph cellularly embedded in a closed surface (orientable or non-orientable, without boundary); when a graph is disconnected, each connected component is embedded in its own surface. Vertices are points and edges are simple curves whose endpoints are vertices (for a loop, the endpoint is a distinguished point of a simple closed curve); these simple curves only meet at endpoints, and when the points and curves representing the vertices and edges of the graph are removed, each of the connected components that remain is homeomorphic to an open disk and corresponds to a *face* of the map. The *degree* of a face is the number of edges traversed when walking around it, any edges that are revisited being counted twice.

The equivalence class of maps up to homeomorphism of the surface is captured well by the definition of a map as a "combinatorial embedding": A *map* is a tuple $(C, \alpha_0, \alpha_1, \alpha_2)$ in which C is a finite set, whose elements are called *crosses*, $\alpha_0, \alpha_1, \alpha_2$ are involutions on C without fixed points, and α_0 and α_2 commute. The tuple $(C, \alpha_0, \alpha_1, \alpha_2)$ is also specified by C, the involutions α_0, α_2 and either the permutation $\tau = \alpha_1\alpha_2$ or the permutation $\phi = \alpha_1\alpha_0$. See Figure 1. Orbits of $\langle \alpha_0, \alpha_2 \rangle$ correspond to edges, to each vertex correspond two orbits of τ (determining the vertex rotation), and to each face correspond two orbits of ϕ; this is the approach taken by Tutte [10, Chapter X].

[1]Email: dgarijo@us.es. Research of D. G. supported by project PID2019-104129GB-I00/AEI/10.13039/501100011033.
[2]Email: andrew@iuuk.mff.cuni.cz. Research of A. G. supported by Czech Science Foundation grant GAČR 19-21082S.
[3]Email: lluis.vena@gmail.com. Research of L. V. supported by the project 18-13685Y of the Czech Science Foundation (GAČR), by the postdoctoral fellowship programme Beatriu de Pinós BP2018, funded by the AGAUR (Government of Catalonia) and by the Horizon 2020 programme of the European Union under the grant agreement No. 801370.

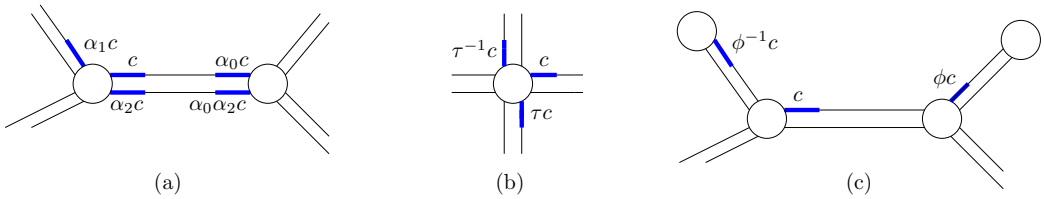


Figure 1: The permutations α_0, α_1 and α_2 defining a map acting on a cross c, (b) vertex permutation $\tau = \alpha_1\alpha_2$, (c) face permutation $\phi = \alpha_1\alpha_0$.

Morphisms between sets with added structure are mappings that preserve this structure; for instance, graph homomorphisms are mappings between vertex sets that preserve adjacency. For multigraphs (loops and parallel edges allowed), a homomorphism is defined as a pair of mappings, one on vertices the other on edges, which together preserve vertex-edge incidences. Maps have several representations, some emphasizing their topological structure (cellular embeddings, ribbon graphs), some emphasizing their combinatorial structure (vertex-edge-face flags, rotation systems, graph-encoded maps, Tutte's permutation axiomatization). What counts as a map homomorphism may thus depend on which representation is chosen – what structure is to be preserved exactly? To start on firm ground, isomorphism of maps has just one candidate for its definition, no matter what representation is chosen. In terms of Tutte's permutation axiomatization, an *isomorphism* between maps is a bijection between their crosses sets that commutes with α_0, α_2 and $\tau = \alpha_1\alpha_2$ (as defined in the respective maps). By dropping the bijective condition, we obtain the definition of map homomorphism given both by Malnič, Nedela, and Škoviera [8] (restricted to the case of orientable surfaces) and by Litjens and Sevenster [7] (restricted to locally bijective mappings, the context being universal covers of graphs). Map homomorphisms defined in this way preserve local combinatorial and topological structure (such as vertex-edge-face incidences and vertex rotations) but do not preserve such global topological parameters as orientability or genus.

We formulate a new definition of map homomorphism that ensures that homomorphisms preserve the surface topology; our definition is mainly based on a vertex identification operation called *vertex gluing*. We define vertex gluing in terms of cross permutations and explain how it can be realized as the inserting of an edge followed by its contraction. This allows us to draw on properties of edge deletion and contraction in order to establish that vertex gluing is the only way to identify vertices in a map while preserving genus and orientability. Map homomorphisms in our sense (Definition 2) preserve not only orientability and genus but also other key topological features (such as the contractibility of facial walks) and the combinatorial structure of the map (when forgetting the embedding, a homomorphism between maps gives a graph homomorphism between their underlying graphs).

Using our definition of map homomorphism, we define a *core* of a map as a minimal homomorphic image, analogously to how the core of a graph is defined. Cores are key to the study of graph homomorphisms [6], and form a poset called the homomorphism order. We establish several properties of map cores shared with graph cores, and characterize map cores in terms of a certain type of contractible closed walk, which, roughly speaking, separate off a disk from the remainder of the map's surface. After giving applications of this characterization, we show that, unlike graph homomorphisms, the map homomorphism order has no dense intervals, and is thus not universal. Finally, we produce examples of maps with an infinite number of cores between them, and an example of an infinite chain of gaps.

The main contribution of our work lies not only in introducing a new notion of map homomorphism and establishing several properties of maps that are cores under this notion of homomorphism, but also in giving a combinatorial formulation for various topological features of a map when viewed as a cellular embedding of a graph in a compact surface. For example, the characterization of cores requires a precise combinatorial characterization of when some closed curves traced out on the map's surface in traversing a closed walk are contractible in the topological sense (a more general notion of contractible is needed than that given in [9], which does not include closed walks that revisit edges within its scope).

2 Map homomorphisms

For multigraphs, a homomorphism is given by a vertex-mapping and an edge-mapping that together preserve vertex-edge incidences. Another way of putting this is that there is a multigraph homomorphism from Γ to Γ' if a subgraph of Γ' can be obtained from Γ by a sequence of vertex identifications followed, if necessary, by the removal of some parallel edges. Associated with this homomorphism is a graph on $V(\Gamma)$ with edges joining pairs of identified vertices by the homomorphism. This graph has no cycles, for otherwise the corresponding sequence of vertex identifications would not always identify distinct vertices. Thus, a graph homomorphism from Γ can be specified by a forest on $V(\Gamma)$, the vertices in a connected component of the forest all being sent to the same vertex by the homomorphism; after performing the vertex identifications, some edges parallel to other edges may be removed.

In order to define a map homomorphism in a similar way, we need to specify how vertices are to be identified so as to produce another map, and what it means for edges of a map to be parallel.

For a graph Γ, the result of identifying vertices u and v can be defined in terms of edge contraction as the graph $(\Gamma + e)/e$, where $e = uv$ and $\Gamma + e = (V, E \cup \{e\})$ (if u, v are already adjacent in Γ then e is parallel to an existing edge in Γ). For a map M embedding Γ in a surface, there is in general more than one way to add an edge e joining distinct vertices u and v so as to produce another map: an unambiguous choice, however, is forced if we stipulate that the map $M + e$ is embedded in the same surface as M (i.e. $M + e$ has the same orientability and genus as M), and then specify not only the pair of vertices to be joined but the face that is subdivided by joining them or the faces that are merged upon joining them. Such a vertex identification with its attendant splitting or merging of faces is what we define as *vertex gluing*. In a similar way to graphs, a sequence of such vertex gluings applied to a map M corresponds to a forest on $V(M)$, vertices in a connected component of the forest all being sent to the same vertex by the homomorphism. While the forest is in itself a plane map, its edges have the property that, taken in the order of the vertex identifications they represent, they can be added one by one to M while preserving genus and orientability. Vertex gluing (i) preserves the surface in which the map is embedded and (ii) commutes with deletion of an edge incident with two distinct vertices and deletion of a bridge; and (iii) permuting the order of a sequence of vertex gluings gives another sequence of vertex gluings, which results in the same map.

Parallel edges are interchangeable when it comes to the existence of graph homomorphisms. There are, however, non-isomorphic maps with the same underlying graph which differ only in the placement of parallel edges. Thus, in order to carry over to maps the property of parallel edges in graphs being indistinguishable for homomorphisms, we need to add a topological constraint: two edges of a map M are *duplicate* if they are incident with a common face of degree two. A refinement of the equivalence relation of being parallel in the underlying graph results: parallel edges in the underlying graph of the map belong to the same class if they can be merged together within the map's surface without passing through a vertex. (Compare how vertices can only be glued together if they can be merged together within the map's surface into one vertex without passing through an edge.) *Edge gluing* a pair of duplicate edges means to identify them, effectively deleting one of the pair. We see that (i) gluing duplicate edges preserves the surface in which the map is embedded, (ii) duplicate edges remain duplicate under vertex gluing, and (iii) gluing duplicates can be done before or after a vertex gluing without changing the resulting map.

The above properties of vertex and edge gluings imply that a sequence of vertex gluings followed by a sequence of duplicate edge gluings takes one map onto another map of the same genus and orientability: the mapping induced on the underlying graphs of the maps is an epimorphism (surjective graph homomorphism). This motivates the following definitions.

Definition 1. An *epimorphism* $n : M \twoheadrightarrow N$ from a map M onto a map N is an isomorphism of N to a map obtained from M by performing a sequence of vertex gluings and a sequence of duplicate edge gluings. A *monomorphism* from M to N is an isomorphism from M to a submap M' of N (N and M of the same orientability and genus).

Definition 2. Let M, M' be maps of the same orientability and genus. A *map homomorphism* from M to M' is a composition of an epimorphism from M onto a submap N of M' and a monomorphism from N into M'.

Homomorphisms are just defined between maps of the same orientability and genus not only because surjective homomorphisms preserve these parameters but also in order that restriction and composition of homomorphisms is well defined.

3 Cores

With the definition of map homomorphism in hand, we define cores analogously to graphs: Given a map M, a submap N of M is a *core* of M if there is a homomorphism from M to N but no homomorphism from M to N' for any proper submap N' of N.

Every map M has a core, and as a homomorphic image of M, it has the same orientability and genus as M. It has no duplicate edges and, in contrast to graphs, cores for map homomorphisms need not to be connected. However, they share other properties with graph cores (for the latter see [6]).

Proposition 3. *The core of a map M is unique (up to isomorphism), and the following properties hold for its core N:*

(i) N is a core (i.e. N is its own core);

(ii) there is a homomorphism from M to N whose restriction to N is the identity;

(iii) N is an induced submap of M after gluing of any duplicate edges;

(iv) given another map M' with core N', there is a homomorphism from M to M' if and only if there is a homomorphism from N to N'.

We now focus on the structural characterization of cores of connected maps, for which we need to define *primitive contractible closed walks*: this is the translation into combinatorial terms of a certain type of contractible curve on the surface in which the underlying graph of the map is embedded.

Definition 4. Let M be a connected map determined by a set of crosses C and permutations $\alpha_0, \alpha_1, \alpha_2$. Let τ and ϕ be its vertex and face permutations, respectively. A cycle κ of not necessarily distinct crosses $(\ c_0 \quad c_1 \quad \cdots \quad c_{\ell-1} \ )$ defines:

- a *closed walk* in M if for each $0 \leq i \leq \ell - 1$ there exists $t_i \geq 0$ such that $c_i = \tau^{t_i} \phi c_{i-1}$ (indices modulo ℓ); the value ℓ is the *length* of κ. (This defines a closed walk in the underlying graph.)
- a *primitive contractible closed walk* in M if there exists a connected submap M' of M with the same orientability and genus as M such that κ is a cycle in the disjoint cycle decomposition of the face permutation of M'.

In terms of the representation of maps as cellular embeddings of multigraphs, the underlying multigraph of the submap M' is cellularly embedded in the same surface Σ as the underlying multigraph of M; cutting along the edges of a primitive contractible closed walk κ, which is a facial walk of M', partitions off a disk Δ from the remainder of the surface Σ such that Δ contains no edges of M'. We say that κ *surrounds* a face f of M if the open disk obtained when cutting Σ along the edges incident with f is contained in Δ.

Theorem 5. *A connected map M is not a core if and only if there exists a primitive contractible closed walk κ in M satisfying the following conditions:*

- *κ is not a face;*

- *surrounds a face whose degree has the same parity and is at least the length of κ;*
- *the remaining faces surrounded by κ have even degree.*

To prove Theorem 5 we analyse the effect of map homomorphisms on primitive contractible closed walks and faces. Briefly, a core N of a connected map M is obtained from M by removing some edges (and isolated vertices). Using that M and its core N have the same orientability and genus and that there is an epimorphism from M to N, we can exactly determine the type of edge that can be removed and how the epimorphism acts on these edges as well as on the faces of M: the latter are mapped onto primitive contractible closed walks of N.

The characterization of map cores yields such results as the following.

Proposition 6. *Let M be a connected map with no vertices of degree one. If all the faces of M have odd degree, then any homomorphism from M to M is an isomorphism.*

Proposition 7. *Let M be a non-empty quasi-tree (a connected map with one face, and at least one vertex). Then M is a core if and only if either M is not plane and the underlying graph has no vertices of degree one, or M is isomorphic to K_2 or K_1.*

Proposition 8. *A plane connected core is either bipartite (and hence isomorphic to K_2), or its underlying graph is a cycle of odd length, or it has at least four faces of odd degree.*

We now turn our attention to the partial order on the set of cores. The existence of a map homomorphism gives a preorder relation on maps of given orientability and genus. By Proposition 3, there is a unique core as a representative in each of the equivalence classes of this preorder, so the cores of maps of given orientability and genus ordered by the existence of a map homomorphism form a poset.

Theorem 9. *The map homomorphism order has the following properties:*

(i) There are no dense intervals.

(ii) There is a pair of cores with infinitely many cores between them.

(iii) There is an infinite chain of cores such that each consecutive pair of the chain forms a gap.

We sketch the main ideas behind our proof of Theorem 9. For part (i), we assume for a contradiction that there is a dense interval, i.e. there are two comparable cores such that, in the interval they define, any pair of cores contains a third one between them. We can prove that among these there is a core M with a homomorphism to another core N such that M and N have the same number of faces of any given odd degree. In a homomorphism from M to N, a face f of even degree in M is mapped to a closed walk in N of even length enclosing some faces, which, by the fact that the number of odd faces in M and N are the same, decomposes into even-length facial walks of a submap of N of the same orientability and genus as N. By Theorem 5, all the faces of N surrounded by the facial walk components of such an even-length closed walk have strictly smaller degree than that of f. But this means that there can only be finitely many cores between M and N.

For part (ii), consider the following family. Let $T_{i,j}$ denote the plane map created from a cycle of length 4 on vertices v_1, v_2, v_3 and v_4 where a path of length i has been added between the vertices v_1 and v_3 and a path of length j has been added from a v_2 to v_4 (each of these paths subdividing one of the faces of the plane cycle C_4). Then we have that $T_{2k+1,2\ell+1}$ is a core for each $k, \ell \geq 0$, and there is a homomorphism from $T_{2k+1,2\ell+1}$ to $T_{2k+1,2(\ell-1)+1}$ for $k \geq 0$ and $\ell \geq 1$ (symmetrically, there is a homomorphism from $T_{2k+1,2\ell+1}$ to $T_{2(k-1)+1,2\ell+1}$ for $k \geq 1$ and $\ell \geq 0$).

An example that establishes part (iii) is the family of odd plane cycles, which forms a chain of gaps. There is a homomorphism from a plane cycle of odd length to one of smaller odd length. If N is a core such that the plane cycle of length $2k+1$ is homomorphic to N and N is homomorphic to the plane cycle of length $2k-1$, then we can show that N is itself one of these two cycles.

4 Concluding remarks

A motivation for formulating a new definition of map homomorphism stems from the connection between the Tutte polynomial of a graph and homomorphism functions to edge-weighted graphs [1] (also known as partition functions of vertex-colouring models). Counting colourings and flows of a graph are important special cases; in fact, any evaluation of the Tutte polynomial expressible as a homomorphism function is the partition function of a Potts model on the graph [2]. For maps, the analogous notions of colourings (or local tensions, rather) and flows (taking non-identity values in a finite group) are counted by evaluations of the surface Tutte polynomial [3, 4], and have been expressed as partition functions of edge-colouring models [7]. This leaves the question of whether these enumerations can be expressed in terms of map homomorphisms as partition functions of vertex-colouring models. To begin to answer this question we needed to formulate a definition of map homomorphism that extends that of graph homomorphism while respecting the topology of the graph embedding.

Regarding the map homomorphism order, we do not know yet if there is an infinite antichain. We are however able to recycle the argument in [6, Theorem 6] to construct arbitrarily large antichains in the poset of plane cores (which happen also to be graph cores) such that all the elements of the antichain are homomorphic to a fixed plane core (which, in contrast to the graph case, has size dependent on the length of the antichain). Finally, Theorem 5 gives a way of recognizing whether a given map is a core, but it does not (at least not directly) yield a polynomial-time algorithm. Recognizing non-bipartite graph cores is NP-complete [6, Theorem 7].

References

[1] M. Freedman, L. Lovász, and A. Schrijver. "Reflection positivity, rank connectivity, and homomorphism of graphs". In: *J. Amer. Math. Soc.* 20.1 (2007), pp. 37–51. DOI: 10.1090/S0894-0347-06-00529-7.

[2] D. Garijo, A. Goodall, and J. Nešetřil. "Distinguishing graphs by their left and right homomorphism profiles". In: *European J. Combin.* 32.7 (2011), pp. 1025–1053. DOI: 10.1016/j.ejc.2011.03.012.

[3] A. Goodall, T. Krajewski, G. Regts, and L. Vena. "A Tutte polynomial for maps". In: *Combin. Probab. Comput.* 27.6 (2018), pp. 913–945. DOI: 10.1017/S0963548318000081.

[4] A. Goodall, B. Litjens, G. Regts, and L. Vena. "A Tutte polynomial for maps II: the non-orientable case". In: *European J. Combin.* 86 (2020), pp. 103095, 32. DOI: 10.1016/j.ejc.2020.103095.

[5] P. Hell and J. Nešetřil. "On the complexity of H-coloring". In: *J. Combin. Theory Ser. B* 48.1 (1990), pp. 92–110. DOI: 10.1016/0095-8956(90)90132-J.

[6] P. Hell and J. Nešetřil. "The core of a graph". In: *Discrete Math.* 109.1-3 (1992). Algebraic graph theory (Leibnitz, 1989), pp. 117–126. DOI: 10.1016/0012-365X(92)90282-K.

[7] B. Litjens and B. Sevenster. "Partition functions and a generalized coloring-flow duality for embedded graphs". In: *J. Graph Theory* 88.2 (2018), pp. 271–283. DOI: 10.1002/jgt.22210.

[8] A. Malnič, R. Nedela, and M. Škoviera. "Regular homomorphisms and regular maps". In: *European J. Combin.* 23.4 (2002), pp. 449–461. DOI: 10.1006/eujc.2002.0568.

[9] B. Mohar and C. Thomassen. *Graphs on surfaces*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 2001, pp. xii+291.

[10] W. T. Tutte. *Graph theory*. Vol. 21. Encyclopedia of Mathematics and its Applications. Reprint of the 1984 original. Cambridge University Press, Cambridge, 2001, pp. xxii+335.

How to build a pillar: a proof of Thomassen's conjecture

Irene Gil Fernández[1*], Hong Liu[2†]

[*]Mathematics Institute and DIMAP, University of Warwick, Coventry, CV4 7AL, UK
[†]Extremal Combinatorics and Probability Group (ECOPRO), Institute for Basic Science (IBS), Daejeon, South Korea.

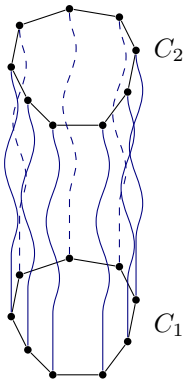
The full version of this work can be found in [7].

Abstract

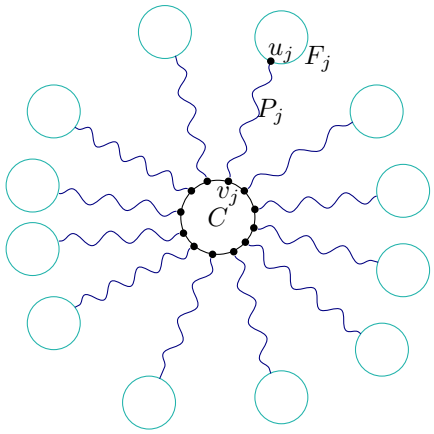
Carsten Thomassen in 1989 conjectured that if a graph has minimum degree more than the number of atoms in the universe ($\delta(G) \geq 10^{10^{10}}$), then it contains a *pillar*, which is a graph that consists of two vertex-disjoint cycles of the same length, s say, along with s vertex-disjoint paths of the same length which connect matching vertices in order around the cycles. Despite the simplicity of the structure of pillars and various developments of powerful embedding methods for paths and cycles in the past three decades, this innocent looking conjecture has seen no progress to date. We give a proof of this conjecture by building a pillar (algorithmically) in sublinear expanders.

1 Introduction

A *pillar* is a graph that consists of two disjoint cycles C_1 and C_2 of the same length, say s, with vertex-sets $V(C_1) = \{v_1, \dots, v_s\}$ and $V(C_2) = \{w_1, \dots, w_s\}$, and s disjoint paths $Q_1, \dots, Q_s$, all of the same length, such that Q_i is a v_i, w_i-path, for each $i \in [s]$; see Figure 1a.



(a) A pillar.



(b) A $(12, s, t)$-kraken.

In 1989, Thomassen [19] conjectured that every graph with sufficiently large constant minimum degree contains a pillar. There have been numerous powerful methods for embedding paths and cycles developed in the past three decades, such as Robertson and Seymour's work on graph linkage [17] (see also [1, 18]), Bondy and Simonovits's use of Breadth First Search [2] (see also [15]), Krivelevich and Sudakov's use of Depth First Search [12] and the use of expanders in a long line of work by Krivelevich

[1]Email: irene.gil-fernandez@warwick.ac.uk.
[2]Email: hongliu@ibs.re.kr. Research of H.L. supported by the Institute for Basic Science (IBS-R029-C4) and the UK Research and Innovation Future Leaders Fellowship MR/S016325/1.

(see e.g. his survey [11] and more recently [4]), see also a recent method developed by Gao, Huo, Liu and Ma [5]. Despite these developments and the simple nature of pillars, the innocent looking conjecture of Thomassen has seen no progress in the past thirty years.

One explanation for the difficulty of this conjecture is the following. Cycles are not hard to embed as all vertices within are of degree 2; and subdivisions, even though having vertices of degree at least 3, can be embedded so that all these high degree vertices are pairwise far apart. Thus, embedding cycles or subdivisions boils down to anchoring at some (well-positioned) vertices and constructing vertex-disjoint paths between pairs of them. On the other hand, in a pillar, degree-3 vertices are jammed into the two cycles C_1 and C_2, which have to be embedded one next to another. To see why degree-3 vertices are game changers, a classical result of Pyber, Rödl and Szemerédi [16] shows that constant average degree does *not* suffice to force a 3-regular subgraph. More precisely, they constructed an n-vertex graph G with average degree $\Omega(\log\log n)$ which contains no r-regular subgraphs for any $r \geq 3$. A priori, it is not clear whether a pillar behaves more like a subdivision or a 3-regular graph.

Our main result confirms Thomassen's conjecture, showing that pillars are fundamentally *different* from 3-regular graphs in the sense that they can be forced by large constant degree.

Theorem 1. *There exists a constant $C > 0$ such that every graph with average degree at least C contains a pillar.*

Our proof can be turned into an algorithm, in which we make crucial use of a notion called sublinear expanders. To deal with the troublesome degree-3 vertices in a pillar, we use a structure called *kraken* (see Definition 9). Roughly speaking, a kraken consists of a cycle, in which every vertex has a large 'boundary'. If we manage to find two krakens, then we can link the matching vertices in their cycles by expanding and connecting their boundaries to obtain a pillar. Finding a *single* kraken in a sublinear expander is already not an easy task; this was done in [6] with an involved argument. To find two krakens here, we prove a *robust* embedding lemma for kraken (Lemma 12), which is the main challenge and contribution of [7]. We expect it to have further applications for embedding problems. Its proof uses the existence of kraken in sublinear expanders from [6] as black box and builds on the techniques developed in the work of Liu and Montgomery [14].

1.1 Notation

For $n \in \mathbb{N}$, let $[n] := \{1, \dots, n\}$. If we claim that a result holds for $0 < a \ll b, c \ll d < 1$, it means that there exist positive functions f, g such that the result holds as long as $a < f(b, c)$ and $b < g(d)$ and $c < g(d)$. We will not compute these functions explicitly. In many cases, we treat large numbers as if they are integers, by omitting floors and ceilings if it does not affect the argument. We write log for the base-e logarithm.

Given a graph G, denote its average degree $2e(G)/|G|$ by $d(G)$, and write $\delta(G)$ and $\Delta(G)$ for its minimum and maximum degree, respectively. Denote the (external) neighbourhood of W by $N(W) = (\cup_{v\in W} N(v)) \setminus W$. We write $N^0_G(W) = W$, and, for each integer $k \geq 1$, let $B^k_G(W) = \cup_{0\leq j\leq k} N^j_G(W)$ the ball of radius k around W in G, that is, the set of all vertices a graph distance at most k to W. We let $B(W) = B^1(W)$.

Let $F \subseteq G$ and H be graphs, and $U \subseteq V(G)$. We write $G[U] \subseteq G$ for the induced subgraph of G on vertex set U. Denote by $G \cup H$ the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, and write $G - U$ for the induced subgraph $G[V(G) \setminus U]$, and $G \setminus F$ for the spanning subgraph of G obtained from removing the edge set of F. For a path P, we write $\ell(P)$ for its length, which is the number of edges in the path. Where we say P is a path from a vertex set A to a disjoint vertex set B, we mean that P has one endvertex in each of A and B, and no internal vertices in $A \cup B$.

2 Sublinear expanders

Our proof makes use of the sublinear expander introduced by Komlós and Szemerédi [10]. We shall use the following extension from [8].

Definition 2. Let $\varepsilon_1 > 0$ and $k > 0$ be two constants. A graph G is an (ε_1, k)-*robust-expander* if for all $X \subset V(G)$ with $k/2 \le |X| \le |G|/2$, and any subgraph $F \subseteq G$ with $e(F) \le d(G) \cdot \varepsilon(|X|)|X|$, we have

$$|N_{G\setminus F}(X)| \ge \varepsilon(|X|) \cdot |X|,$$

where

$$\varepsilon(x) = \varepsilon\,(x, \varepsilon_1, k) = \begin{cases} 0 & \text{if } x < k/5, \\ \varepsilon_1/\log^2(15x/k) & \text{if } x \ge k/5. \end{cases}$$

We invoke [8, Lemma 3.2], which asserts that every graph contains a robust expander subgraph with almost the same average degree, to reduce Theorem 1 to an expander. One key consequence of the expansion is the so-called short diameter property. That is, we can find short paths robustly between two sufficiently large sets.

Lemma 3 ([14], Lemma 3.4)**.** *For each $0 < \varepsilon_1, \varepsilon_2 < 1$, there exists $d_0 = d_0(\varepsilon_1, \varepsilon_2)$ such that the following holds for each $n \ge d \ge d_0$ and $x \ge 1$. Let G be an n-vertex $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \ge d - 1$. Let $A, B \subseteq V(G)$ with $|A|, |B| \ge x$, and let $W \subseteq V(G) \setminus (A \cup B)$ satisfy $|W| \log^3 n \le 10x$. Then, there is a path from A to B in $G - W$ with length at most $\frac{40}{\varepsilon_1} \log^3 n$.*

We make use of the following three results regarding robust expansion of sets. For the first one, which enables us to grow a set A past some given set X as long as X does not interfere with each sphere around A too much, we need the following definition.

Definition 4. For $\lambda > 0$ and $k \in \mathbb{N}$, we say that a vertex set X in a graph G is (λ, k)-*thin around* A if $X \cap A = \varnothing$ and, for each $i \in \mathbb{N}$,

$$|N_G(B^{i-1}_{G-X}(A)) \cap X| \le \lambda \cdot i^k.$$

Proposition 5 ([6], Proposition 2.5)**.** *Let $0 < 1/d \ll \varepsilon_1 \ll 1/\lambda, 1/k$ and $1 \le r \le \log n$. Suppose G is an n-vertex $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \ge d$, and X, Y are sets of vertices with $|X| \ge 1$ and $|Y| \le \frac{1}{4}\varepsilon(|X|) \cdot |X|$. Let W be a (λ, k)-thin set around X in $G - Y$. Then, for each $1 \le r \le \log n$, we have*

$$|B^r_{G-W-Y}(X)| \ge \exp(r^{1/4}).$$

On the other hand, when we are given a large collection of sets in a sublinear expander, we can use the following lemma to find one set within the collection that expands robustly to medium (polylogarithmic) size.

Lemma 6 ([14], Lemma 3.7)**.** *For each $0 < \varepsilon_1 < 1$, $0 < \varepsilon_2 < 1/5$ and $k \in \mathbb{N}$, there exists $d_0 = d_0(\varepsilon_1, \varepsilon_2, k)$ such that the following holds for each $n \ge d \ge d_0$. Suppose that G is an n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \ge d$. Let $U \subseteq V(G)$ satisfy $|U| \le \exp((\log\log n)^2)$. Let $r \ge n^{1/8}$ and $\ell_0 = (\log\log n)^{20}$. Suppose (A_i, B_i, C_i), $i \in [r]$, are such that the following hold for each $i \in [r]$.*

A1 $|A_i| \ge d_0$.

A2 *$B_i \cup C_i$ and A_i are disjoint sets in $V(G) \setminus U$, with $|B_i| \le |A_i|/\log^{10}|A_i|$.*

A3 *C_i is $(\sqrt{|A_i|}, 1)$-thin around A_i in $G - U - B_i$.*

A4 *Each vertex in $B^{\ell_0}_{G-U-B_i-C_i}(A_i)$ has at most $d/2$ neighbours in U.*

A5 *For each $j \in [r] \setminus \{i\}$, A_i and A_j are at least a distance $2\ell_0$ apart in $G - U - B_i - C_i - B_j - C_j$.*

Then, for some $i \in [r]$,

$$|B^{\ell_0}_{G-U-B_i-C_i}(A_i)| \geq \log^k n.$$

The third one allows us to find a linear size vertex set with polylogarithmic diameter in G while avoiding an arbitrary set of size $o(n/\log^2 n)$.

Lemma 7 ([14], Lemma 3.12)**.** *Let $0 < 1/d \ll \varepsilon_1, \varepsilon_2 < 1$ and let G be an n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$. For any $W \subseteq V(G)$ with $|W| \leq \varepsilon_1 n/100 \log^2 n$, there is a set $B \subseteq G - W$ with size at least $n/25$ and diameter at most $200\varepsilon_1^{-1} \log^3 n$.*

3 Krakens and adjusters

To define one of the main building objects of a pillar we need to introduce a basic structure that we often use, which consists of a large set with small radius.

Definition 8. Given a vertex v in a graph F, F is a (D, m)-*expansion of* v if $|F| = D$ and every vertex of F is a distance at most m from v.

Definition 9. For $k, s, t \in \mathbb{N}$, a (k, s, t)-*kraken* is a graph that consists of a cycle C with vertices $v_1, \ldots, v_k$, vertices $u_1, \ldots, u_k$ out of $V(C)$, and subgraphs F_j and P_j, $j \in [k]$, such that

- $\{F_j : j \in [k]\}$ is a collection of sets disjoint from each other and from $V(C)$, and each F_j is a (t, s)-expansion of u_j. We call each F_j a *leg* and u_j its *end*.
- $\{P_j : j \in [k]\}$ is a collection of pairwise disjoint paths, and each P_j is a v_j, u_j-path of length at most $10s$ with internal vertices disjoint from $V(C) \cup (\cup_{i \in [k]} V(F_i))$.

We usually write a kraken as a tuple (C, u_j, F_j, P_j), $j \in [k]$ (see Figure 1b). The following result guarantees a large kraken in a sublinear expander.

Lemma 10 ([6], Lemma 3.2)**.** *Let $0 < 1/d \ll \varepsilon_1, \varepsilon_2, 1/b < 1$. Let G be an n-vertex $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$. Let $m = 200\varepsilon_1^{-1} \log^3 n$. Then, there exists a $(k, m, \log^b n)$-kraken (C, u_j, F_j, P_j), $j \in [k]$, in G for some $k \leq \log n$.*

Another important tool we need in our proof is a recent lemma of Liu and Montgomery [14] (Lemma 11), which robustly finds paths of specific lengths between a given pair of vertices in a sublinear expander with some mild conditions.

The key object involved in that result is called *adjuster*, and it consists of an even cycle C, together with two disjoint large connected subgraphs F_1, F_2 attached to two almost-antipodal vertices v_1, v_2 on the cycle C. If C has length 2ℓ, for some $\ell \leq \log n$, there are two v_1, v_2-paths, one of length $\ell + 1$ and the other with length $\ell - 1$. The subgraphs F_1, F_2 are set to be comfortably larger than the size of C, so that they can be connected by a short path while avoiding C. The idea is to link many such structures sequentially to form an *adjuster*, so that we can use it to find paths of many different lengths by varying the length of the path we take around each cycle.

To record the parity of paths between two vertices in a connected bipartite graph, for any connected bipartite graph H and $u, v \in V(H)$, define

$$\pi(u, v, H) = \begin{cases} 0 & \text{if } u \text{ and } v \text{ are in the same vertex class in the (unique) bipartition of } H, \\ 1 & \text{if } u \text{ and } v \text{ are in different vertex classes in the bipartition of } H. \end{cases}$$

Lemma 11 ([14], Lemma 4.8)**.** *There exists some $\varepsilon_1 > 0$ such that, for any $0 < \varepsilon_2 < 1/5$ and $b \geq 10$, there exists $d_0 = d_0(\varepsilon_1, \varepsilon_2, b)$ such that the following holds for each $n \geq d \geq d_0$. Suppose that G is an n-vertex Q_3-free bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$.*

Suppose $\log^{10} n \leq D \leq \log^b n$, and $U \subseteq V(G)$ with $|U| \leq D/2 \log^3 n$, and let $m = \frac{8000}{\varepsilon_1} \log^3 n$. Suppose $F_1, F_2 \subseteq G - U$ are vertex-disjoint such that F_i is a (D, m)-expansion of v_i, for each $i \in [2]$. Let $\log^7 n \leq \ell \leq n/\log^{10} n$ be such that $\ell = \pi(v_1, v_2, G) \mod 2$.

Then, there is a v_1, v_2-path with length ℓ in $G - U$.

4 Proof idea of Theorem 1

To prove Theorem 1, we first find two copies of krakens whose cycles are of the same length. Then we link sequentially and disjointly the legs of one kraken to those of the other one. We package these two steps into the following two lemmas, respectively.

The first lemma is the key one, which constructs a kraken robustly in an expander.

Lemma 12. *For each $0 < \varepsilon_1, \varepsilon_2 < 1$ and integer $b \geq 10$, there exists $d_0 = d_0(\varepsilon_1, \varepsilon_2, b)$ such that the following holds for each $n \geq d \geq d_0$.*

Let G be a Q_3-free n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$. Let L be the set of vertices with degree at least $e^{(\log\log n)^2}$. Let $m = 200\varepsilon^{-1}\log^3 n$ and let $U \subseteq V(G)$ satisfy $|U| \leq (\log n)^{2b}$.

Then, for some $k \leq \log n$, $G - U$ contains a $(k, 2m, (\log n)^b)$-kraken (C, u_j, F_j, P_j), $j \in [k]$, such that

- *for each $j \in [k]$, either $u_j \in L$ or $F_j \subseteq G - L$; and*
- *any distinct legs $F_j, F_{j'}$ in $G - L$ are a distance at least $(\log n)^{1/10}$ apart from each other and from $U \setminus L$ in $G - L$.*

The second lemma allows us to link each pair of vertices in the cycles inside krakens via their legs disjointly to construct a pillar. Lemma 11 kicks in here to make sure that all the paths used are of the same length.

Lemma 13. *For each $0 < \varepsilon_1, \varepsilon_2 < 1$ and $t \geq 10$, there exists $d_0 = d_0(\varepsilon_1, \varepsilon_2, t)$ such that the following holds for each $n \geq d \geq d_0$.*

Let G be a Q_3-free n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$ and L be the set of vertices with degree at least $e^{(\log\log n)^2}$. Let $m = 400\varepsilon^{-1}\log^3 n$ and let $\mathsf{K}_\alpha = (C_\alpha, u_j^\alpha, F_j^\alpha, P_j^\alpha)$ and $\mathsf{K}_\beta = (C_\beta, u_j^\beta, F_j^\beta, P_j^\beta)$, $j \in [s]$, be two disjoint $(s, m, (\log n)^{2t})$-krakens, for some $s \leq \log n$, with $V(C_\alpha) = \{v_1^\alpha, \ldots, v_s^\alpha\}$ and $V(C_\beta) = \{v_1^\alpha, \ldots, v_s^\alpha\}$, and such that

- *for each $\sigma \in \{\alpha, \beta\}$ and $j \in [s]$, either $u_j^\sigma \in L$ or $F_j^\sigma \subseteq G - L$; and*
- *all legs in K_α and K_β lying completely in $G - L$ are a distance at least $(\log n)^{1/10}$ apart from each other in $G - L$.*

Then, for any $\log^7 n \leq \ell \leq \log^t n$ with $\ell = \pi(v_1^\alpha, v_1^\beta, G)$, there is a collection of pairwise disjoint paths Q_i, $i \in [s]$, such that each Q_i is a v_i^α, v_i^β-path of length ℓ internally disjoint from C_α and C_β.

References

[1] B. Bollobás and A. Thomason. "Highly linked graphs". In: *Combinatorica* 16.3 (1996), pp. 313–320. DOI: 10.1007/BF01261316.

[2] J. A. Bondy and M. Simonovits. "Cycles of even length in graphs". In: *J. Combinatorial Theory Ser. B* 16 (1974), pp. 97–105. DOI: 10.1016/0095-8956(74)90052-5.

[3] K. Corradi and A. Hajnal. "On the maximal number of independent circuits in a graph". In: *Acta Math. Acad. Sci. Hungar.* 14 (1963), pp. 423–439. DOI: 10.1007/BF01895727.

[4] J. Erde, M. Kang, and M. Krivelevich. *Expansion in supercritical random subgraphs of the hypercube and its consequences*. Preprint. 2021. DOI: 10.48550/ARXIV.2111.06752.

[5] J. Gao, Q. Huo, C.-H. Liu, and J. Ma. "A Unified Proof of Conjectures on Cycle Lengths in Graphs". In: *Int. Math. Res. Not. IMRN* 10 (2022), pp. 7615–7653. DOI: 10.1093/imrn/rnaa324.

[6] I. Gil Fernández, J. Kim, Y. Kim, and H. Liu. "Nested cycles with no geometric crossings". In: *Proc. Amer. Math. Soc. Ser. B* 9 (2022), pp. 22–32. DOI: 10.1090/bproc/107.

[7] I. Gil Fernández and H. Liu. *How to build a pillar: a proof of Thomassen's conjecture*. Preprint. 2022. DOI: 10.48550/ARXIV.2201.07777.

[8] J. Haslegrave, J. Kim, and H. Liu. "Extremal density for sparse minors and subdivisions". In: *Int. Math. Res. Not. IMRN* (2021). To appear. DOI: 10.1093/imrn/rnab154.

[9] J. Kim, H. Liu, M. Sharifzadeh, and K. Staden. "Proof of Komlós's conjecture on Hamiltonian subsets". In: *Proc. Lond. Math. Soc. (3)* 115.5 (2017), pp. 974–1013. DOI: 10.1112/plms.12059.

[10] J. Komlós and E. Szemerédi. "Topological cliques in graphs. II". In: *Combin. Probab. Comput.* 5.1 (1996), pp. 79–90. DOI: 10.1017/S096354830000184X.

[11] M. Krivelevich. "Expanders - how to find them, and what to find in them". In: *Surveys in combinatorics 2019*. Ed. by A. Lo, R. Mycroft, G. Perarnau, and A. Treglown. Vol. 456. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2019, pp. 115–142.

[12] M. Krivelevich and B. Sudakov. "The phase transition in random graphs: a simple proof". In: *Random Structures & Algorithms* 43.2 (2013), pp. 131–138. DOI: 10.1002/rsa.20470.

[13] H. Liu and R. H. Montgomery. "A proof of Mader's conjecture on large clique subdivisions in C_4-free graphs". In: *J. Lond. Math. Soc. (2)* 95.1 (2017), pp. 203–222. DOI: 10.1112/jlms.12019.

[14] H. Liu and R. H. Montgomery. *A solution to Erdős and Hajnal's odd cycle problem*. Preprint. 2020. DOI: 10.48550/ARXIV.2010.15802.

[15] O. Pikhurko. "A note on the Turán function of even cycles". In: *Proc. Amer. Math. Soc.* 140.11 (2012), pp. 3687–3692. DOI: 10.1090/S0002-9939-2012-11274-2.

[16] L. Pyber, V. Rödl, and E. Szemerédi. "Dense graphs without 3-regular subgraphs". In: *J. Combin. Theory Ser. B* 63.1 (1995), pp. 41–54. DOI: 10.1006/jctb.1995.1004.

[17] N. Robertson and P. D. Seymour. "Graph minors. XIII. The disjoint paths problem". In: *J. Combin. Theory Ser. B* 63.1 (1995), pp. 65–110. DOI: 10.1006/jctb.1995.1006.

[18] R. Thomas and P. Wollan. "An improved linear edge bound for graph linkages". In: *European J. Combin.* 26.3-4 (2005), pp. 309–324. DOI: 10.1016/j.ejc.2004.02.013.

[19] C. Thomassen. "Configurations in graphs of large minimum degree, connectivity, or chromatic number". In: *Combinatorial Mathematics: Proceedings of the Third International Conference (New York, 1985)*. Vol. 555. 1. New York Acad. Sci., New York, 1989, pp. 402–412. DOI: 10.1111/j.1749-6632.1989.tb22479.x.

Rainbow solutions of a linear equation with coefficients in $\mathbb{Z}/p\mathbb{Z}$

Mario Huicochea[1]

CONACyT-UAZ, Zacatecas, México

Abstract

Let p be a prime, $n \in \mathbb{Z}^+$ and $w \in (0,1)$. Given a colouring $\chi : \mathbb{Z}/p\mathbb{Z} \to \{1,2,\ldots,n\}$ and a linear equation

$$\mathcal{L}: \qquad a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

with $a_1, a_2, \ldots, a_n \in (\mathbb{Z}/p\mathbb{Z})^*$ and $b \in \mathbb{Z}/p\mathbb{Z}$ fixed elements, we denote by $R(\chi, \mathcal{L})$ the family of vectors $(b_1, b_2, \ldots, b_n) \in (\mathbb{Z}/p\mathbb{Z})^n$ such that $a_1b_1 + a_2b_2 + \ldots + a_nb_n = b$ and $\chi^{-1}(i) \cap \{b_1, b_2, \ldots, b_n\} \neq \emptyset$ for each $i \in \{1,2,\ldots,n\}$. Here the main result is that there exists a constant $c = c(w,n) > 0$ with the following property: if χ is such that $\min_{1\leq i\leq n} |\chi^{-1}(i)| \geq wp + 1$ and if not all the coefficients $a_1, a_2, \ldots, a_n$ of $\mathcal{L}$ are equal, then

$$|R(\chi, \mathcal{L})| \geq cp^{n-1}.$$

Moreover, this statement is sharp in different directions. A result about the solutions of $\mathcal{L}$ in a grid is used in its proof and it is interesting in its own right.

1 Introduction

Here $\mathbb{R}, \mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}_0^+$ will denote the set of real numbers, integers, positive integers and nonnegative integers, respectively. Also, for any prime p, we will we denote by $\mathbb{Z}/p\mathbb{Z}$ the congruence classes modulo p with its usual field structure. For any $m \in \mathbb{Z}$, we will denote by $\overline{m}$ its projection in $\mathbb{Z}/p\mathbb{Z}$. We set $(\mathbb{Z}/p\mathbb{Z})^* := \mathbb{Z}/p\mathbb{Z} \setminus \{\overline{0}\}$. Now, for a linear equation

$$\mathcal{L}: \qquad a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

with $b, a_1, a_2, \ldots, a_n \in \mathbb{Z}/p\mathbb{Z}$, we write

$$\mathcal{Z}(\mathcal{L}) := \{(b_1, b_2, \ldots, b_n) \in (\mathbb{Z}/p\mathbb{Z})^n : \ a_1b_1 + a_2b_2 + \ldots + a_nb_n = b\}.$$

The elements $a_1, a_2, \ldots, a_n$ will be called the *coefficients* of $\mathcal{L}$. For any $n \in \mathbb{Z}^+$, we denote by $\mathbb{S}_n$ the set of permutations of $\{1,2,\ldots,n\}$.

Let $n \in \mathbb{Z}^+$ and X be a set. An *n-colouring* of X is a map $\chi : X \to \{1,2,\ldots,n\}$. The elements i of $\{1,2,\ldots,n\}$ will be called *colours* while the preimages $\chi^{-1}(i)$ will be called *chromatic classes* with respect to χ. A subset Y of X is *rainbow* with respect to χ if it intersects each chromatic class; if it is clear with respect to which colouring we are talking about, we simply say that Y is rainbow. We say that a subset Y of X is *monochromatic* with respect to χ if Y is contained in a chromatic class. Here we will be mainly interested in the colourings of $\mathbb{Z}/p\mathbb{Z}$. For an n-colouring χ of $\mathbb{Z}/p\mathbb{Z}$ and a linear equation

$$\mathcal{L}: \qquad a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

with $b, a_1, a_2, \ldots, a_n \in \mathbb{Z}/p\mathbb{Z}$, we denote by $R(\chi, \mathcal{L})$ the set of vectors $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathcal{Z}(\mathcal{L})$ such that $\{b_1, b_2, \ldots, b_n\}$ is rainbow. The elements $\mathbf{b}$ of $R(\chi, \mathcal{L})$ will be called *rainbow solutions* of $\mathcal{L}$ with

[1]Email: dym@cimat.mx

respect to χ; if it is clear with respect to which linear equation and colouring we are talking about, we simply call $\mathbf{b}$ a rainbow solution.

A classical problem in discrete mathematics is to find a lower bound on the number of monochromatic solutions of linear equations over $\mathbb{Z}/p\mathbb{Z}$ and a number of results in this direction can be found nowadays, see for instance [2, 3, 12]. However, less is known about the number of rainbow solutions of linear equations over a field. The first step was to know for which linear equations with n variables and coefficients in $\mathcal{L}$ and n-colourings χ of $\mathbb{Z}/p\mathbb{Z}$ can we warranty that $R(\chi,\mathcal{L})$ is not empty. In the case $n=3$, very precise descriptions of the equations and colouring with this property can be found in the literature, see [7, 8, 9, 10]. For $n>3$, D. Conlon was able to prove that for any equation $a_1x_1+a_2x_2+\ldots+a_nx_n=b$ with $b\in\mathbb{Z}/p\mathbb{Z}$ and $a_1,a_2,\ldots,a_n\in(\mathbb{Z}/p\mathbb{Z})^*$ and not all the coefficients equal, each n-colouring χ of $\mathbb{Z}/p\mathbb{Z}$ such that chromatic class has at least n elements satisfies that $R(\chi,\mathcal{L})\neq\emptyset$, see [4, Thm.1]. Later this result was improved.

Theorem 1. *Let p be a prime, $n\in\mathbb{Z}$ with $n\geq 4$ and a linear equation*

$$\mathcal{L}: \qquad a_1x_1+a_2x_2+\ldots+a_nx_n=b$$

with $b\in\mathbb{Z}/p\mathbb{Z}$ and $a_1,a_2,\ldots,a_n\in(\mathbb{Z}/p\mathbb{Z})^*$ such that $a_i\neq a_j$ for some $i,j\in\{1,2,\ldots,n\}$. For any n-colouring χ of $\mathbb{Z}/p\mathbb{Z}$ such that $\min_{1\leq i\leq n}|\chi^{-1}(i)|\geq 4$, we get that

$$R(\chi,\mathcal{L})\neq\emptyset.$$

Proof. See [5, Thm.1.5]. □

The next step was to know not only if there were solutions but the number of them. For the linear equation

$$\mathcal{L}_0: \qquad x_1+x_2-\overline{2}x_3=\overline{0},$$

Balandraud in [1, Prop. 1] and Montejano and Serra in [11, Prop. 11] found lower bounds of $|R(\chi,\mathcal{L}_0)|$ for some colourings χ; also some progress was done for more general linear equations and colourings in [6]. However, the problem of determining nontrivial lower bounds of $|R(\chi,\mathcal{L})|$ remained open and it is the main motivation of this paper. The first result of this paper is the next one.

Theorem 2. *Let p be a prime, $n\in\mathbb{Z}$ with $n\geq 3$, $w\in\left(\left(\frac{10^{100}\cdot 4(n-3)}{p}\right)^{\frac{1}{3}},1\right]$ and a linear equation*

$$\mathcal{L}: \qquad a_1x_1+a_2x_2+\ldots+a_nx_n=b$$

with $b\in\mathbb{Z}/p\mathbb{Z}$ and $a_1,a_2,\ldots,a_n\in(\mathbb{Z}/p\mathbb{Z})^*$ such that $a_i\neq a_j$ for some $i,j\in\{1,2,\ldots,n\}$. For any n-colouring χ of $\mathbb{Z}/p\mathbb{Z}$ such that $\min_{1\leq i\leq n}|\chi^{-1}(i)|\geq wp+1$, we get that

$$|R(\chi,\mathcal{L})|\geq\left(\frac{w^{19}}{10^{1600}}\right)^{n-1}p^{n-1}.$$

Three remarks about Theorem 2 can be done.

1) The size of $|R(\chi,\mathcal{L})|$ depends on the size of the smallest chromatic classes. Indeed, assume without loss of generality that $|\chi^{-1}(1)|\leq|\chi^{-1}(2)|\leq\ldots\leq|\chi^{-1}(n)|$. For each $\mathbf{b}\in R(\chi,\mathcal{L})$, the entries which are in $\chi^{-1}(1),\chi^{-1}(2),\ldots,\chi^{-1}(n-1)$ determine the entry in $\chi^{-1}(n)$. Thus, considering the permutations, we conclude that

$$|R(\chi,\mathcal{L})|\leq n!\prod_{i=1}^{n-1}|\chi^{-1}(i)|. \tag{1}$$

Moreover, as it is already noted in [7], there exist linear equations with two distinct coefficients and colourings χ with $\min_{1\leq i\leq 3}|\chi^{-1}(i)|=1$ such that $R(\chi,\mathcal{L})=\emptyset$. Thus, as a consequence of (1), we have that Theorem 2 leads to $|R(\chi,\mathcal{L})|=\Theta_w(p^{n-1})$.

2) It seems that the coefficient $c(w,n)=\left(\frac{w^{19}}{10^{1600}}\right)^{n-1}$ in Theorem 2 can be improved and it looks like a nice problem to find the best bound.

3) Theorem 2 cannot be extended to linear equations $\mathcal{L}$ where $a_1=a_2=\ldots=a_n$ since there are n-colourings χ of $\mathbb{Z}/p\mathbb{Z}$ such that $R(\chi,\mathcal{L})=\emptyset$, see [4, Sec.4].

Let $n\in\mathbb{Z}^+$ and a linear equation

$$\mathcal{L}: \qquad a_1x_1+a_2x_2+\ldots+a_nx_n=b$$

with $b,a_1,a_2,\ldots,a_n\in\mathbb{Z}/p\mathbb{Z}$. For any ordered family $\mathcal{F}=(B_1,B_2,\ldots,B_n)$ of nonempty subsets $B_1,B_2,\ldots,B_n$ of $\mathbb{Z}/p\mathbb{Z}$, we write

$$R_{\mathcal{F}}(\mathcal{L}):=\mathcal{Z}(\mathcal{L})\cap(B_1\times B_2\times\ldots\times B_n).$$

An important part of the proof of Theorem 2 relies on the next theorem which is interesting in its own right.

Theorem 3. *Let p be a prime, $n\in\mathbb{Z}$ with $n\geq 2$, $w\in\left(\left(\frac{10^{100}\cdot 4(n-3)}{p}\right)^{\frac{1}{3}},\frac{1}{n}\right]$, $B_1,B_2,\ldots,B_n$ be nonempty subsets of $\mathbb{Z}/p\mathbb{Z}$ and a linear equation*

$$\mathcal{L}: \qquad a_1x_1+a_2x_2+\ldots+a_nx_n=b$$

with $b\in\mathbb{Z}/p\mathbb{Z}$ and $a_1,a_2,\ldots,a_n\in(\mathbb{Z}/p\mathbb{Z})^*$ such that $a_1\notin\{\pm a_2\}$. Write $c:=\left(\frac{w^{19}}{10^{1600}}\right)^{n-1}$, $\mathcal{F}_1:=(B_1,B_2,B_3,B_4,\ldots,B_n)$ and $\mathcal{F}_2:=(B_2,B_1,B_3,B_4,\ldots,B_n)$. If $\sum_{i=1}^n|B_i|\geq p$ and $\min_{1\leq i\leq n}|B_i|\geq wp+1$, then

$$\max\{|R_{\mathcal{F}_1}(\mathcal{L})|,|R_{\mathcal{F}_2}(\mathcal{L})|\}\geq cp^{n-1}.$$

An important case of Theorem 2 is when the coefficients in

$$\mathcal{L}: \qquad a_1x_1+a_2x_2+\ldots+a_nx_n=b$$

satisfy that $a_i\in\{\pm a_j\}$ for all $i,j\in\{1,2,\ldots,n\}$, and it needs some effort to solve this special case. However, when there are coefficients a_i,a_j such that $a_i\notin\{\pm a_j\}$, it is not difficult to note that Theorem 3 yields Theorem 2. Therefore we think it is better to spend the last part with a sketch of the proof of Theorem 3.

2 Sketch of the proof of Theorem 3

To begin this sketch, we need some notation. For any $A,A_1,A_2,\ldots,A_n$ nonempty subsets of $\mathbb{Z}/p\mathbb{Z}$, $d,e\in\mathbb{Z}/p\mathbb{Z}$ and $t\geq 0$, set

$$\begin{aligned}
A_1+A_2+\ldots+A_n&:=\{a_1+a_2+\ldots+a_n:\ a_i\in A_i \text{ for all } i\in\{1,2,\ldots,n\}\}\\
r_{A_1,A_2,\ldots,A_n}(d)&:=\{(a_1,a_2,\ldots,a_n)\in A_1\times A_2\times\ldots\times A_n:\ a_1+\ldots+a_n=d\}\\
E_t(A_1,A_2,\ldots,A_n)&:=\{e\in\mathbb{Z}/p\mathbb{Z}:\ r_{A_1,A_2,\ldots,A_n}(e)\geq t\}\\
S_t(A_1,A_2,\ldots,A_n)&:=\{(a_1,\ldots,a_n)\in(\mathbb{Z}/p\mathbb{Z})^n:a_1+\ldots+a_n\in E_t(A_1,\ldots,A_n)\}\\
eA+d&:=\{ea+d:\ a\in A\}.
\end{aligned}$$

For any nonempty subset A of $\mathbb{Z}/p\mathbb{Z}$, denote by $l_d(A)$ the length of the shortest arithmetic progression with difference d which contains A.

Now we start with the sketch. Set $B := a_1B_2, A := E_{(\frac{w^{19}}{10^{1600}}p)^{n-3}}(a_3B_3, a_4B_4, \ldots, a_nB_n)$ and $a := a_1^{-1}a_2$. An important part of the proof is to show that

$$\max\left\{\left|E_{\frac{w^{19}}{10^{1600}}p}(B,A)\right|, \left|E_{\frac{w^{19}}{10^{1600}}p}(aB,A)\right|\right\} > \min\left\{|B|+|A|\left(1+\frac{w^2}{10^{100}}\right), p\left(1-2^{\frac{1}{2}}\frac{w^4}{10^{400}}\right)\right\}. \quad (2)$$

The proof of (2) is done by contradiction. Assume that

$$\max\left\{\left|E_{\frac{w^{19}}{10^{1600}}p}(B,A)\right|, \left|E_{\frac{w^{19}}{10^{1600}}p}(aB,A)\right|\right\} \leq \min\left\{|B|+|A|\left(1+\frac{w^2}{10^{100}}\right), p\left(1-2^{\frac{1}{2}}\frac{w^4}{10^{400}}\right)\right\}. \quad (3)$$

The contradiction is achieved using the next two statements. Lemma 4 is a Balog-Szemerédi-Gowers-type result and it is a crucial part of the proof of Theorem 3.

Lemma 4. *Let p be prime, $w \in (0,1]$, $u \in \left[\frac{1}{p}, \min\left\{\frac{w^{16}}{2^{112}}, \left(\frac{31}{10^{1551}}\right)^{32}\right\}\right]$, $a \in (\mathbb{Z}/p\mathbb{Z})^*$ and B, A be subsets of $\mathbb{Z}/p\mathbb{Z}$ such that $|A|, |B| \geq wp+1$ and $|B| \leq \left(\frac{1}{2}-10u^{\frac{1}{32}}\right)p-2$. Assume that*

$$\left|E_{uw^3p}(B,A)\right|, \left|E_{uw^3p}(aB,A)\right| \leq \min\left\{|B|+|A|\left(1+u^{\frac{1}{16}}w\right), p\left(1-2^{\frac{1}{2}}u^{\frac{1}{4}}\right)\right\}.$$

Then there are B' a subset of B and $d \in (\mathbb{Z}/p\mathbb{Z})^*$ such that

$$|B'| \geq \left(1-2u^{\frac{1}{2}}\right)|B| \geq |B| - 2u^{\frac{1}{2}}p$$

and

$$l_d(B'), l_{a^{-1}d}(B') \leq |B'| + 10u^{\frac{1}{32}}p - 4.$$

Theorem 5. *Let p be prime, $d_1, d_2 \in (\mathbb{Z}/p\mathbb{Z})^*$, $r \in \mathbb{Z}_0^+$ and A be a subset of $\mathbb{Z}/p\mathbb{Z}$. Assume that*

- $l_{d_1}(A) \leq |A| + r$
- $l_{d_2}(A) \leq |A| + r$
- $r+3 \leq |A| \leq p - 4r - 10$.

Then $d_1 \in \{\pm d_2\}$.

On the one hand, we can apply Lemma 4 by (3) so we obtain a big subset B' of B such that $l_d(B'), l_{a^{-1}d}(B') \leq |B'| + 10u^{\frac{1}{32}}p - 4$. On the other hand, we can apply Theorem 5 to $l_d(B')$ and $l_{a^{-1}d}(B')$, however it implies that $a^{-1} \in \{\pm 1\}$ (i.e. $a_1 \in \{\pm a_2\}$) which is impossible by assumption (Theorem 5 roughly establishes that a medium-size set cannot be contained in two short arithmetic progressions with differences $d_1, d_2 \in (\mathbb{Z}/p\mathbb{Z})^*$ such that $d_1 \notin \{\pm d_2\}$). This proves (2) and from now on we assume, without loss of generality that,

$$\left|E_{\frac{w^{19}}{10^{1600}}p}(aB,A)\right| > \min\left\{|B|+|A|\left(1+\frac{w^2}{10^{100}}\right), p\left(1-2^{\frac{1}{2}}\frac{w^4}{10^{400}}\right)\right\}. \quad (4)$$

Write $E := E_{\left(\frac{w^{19}}{10^{1600}}p\right)^{n-2}}(a_2B_2, a_3B_3, \ldots, a_nB_n)$. As a consequence of (4) and an easy counting argument, we get that

$$|E| \geq \left|E_{\frac{w^{19}}{10^{1600}}p}(aB,A)\right| > \min\left\{|B|+|A|\left(1+\frac{w^2}{10^{100}}\right), p\left(1-2^{\frac{1}{2}}\frac{w^4}{10^{400}}\right)\right\}. \quad (5)$$

However, the size of $A = E_{(\frac{w^{19}}{10^{1600}}p)^{n-3}}(a_3B_3, a_4B_4, \ldots, a_nB_n)$ can be bounded below in terms of $|B_3| + |B_4| + \ldots + |B_n|$ using a Cauchy-Davenport-type theorem.

Lemma 6. *Let p be prime, $n \in \mathbb{Z}^+$, $w \in \left[0, \frac{2}{3}\right)$, $u \in \left[0, \min\left\{\frac{w^2}{4}, \frac{1}{n^2}\right\}\right)$ and $A_1, A_2, \ldots, A_n$ be nonempty subsets of $\mathbb{Z}/p\mathbb{Z}$ such that $\min_{1\le i\le n} |A_i| \ge wp + 1$. Then*

$$\left|E_{(uw^2p)^{n-1}}(A_1, \ldots, A_n)\right| \ge \min\left\{p, (|A_1| + \ldots + |A_n|)\left(1 - (n-1)u^{\frac{1}{2}}\right) - n + 1\right\}.$$

Using Lemma 6 and (5), we can obtain that

$$|E| \ge \min\left\{p, |B_2| + |B_3| + \ldots + |B_n| + \frac{(n-3)w^3}{10^{101}}p\right\}$$

so

$$|(-b + a_1B_1) \cap E| \ge \min\left\{|B_1| + |B_2| + \ldots + |B_n| + \frac{(n-3)w^3}{10^{101}}p, p\left(1 - 2^{\frac{1}{2}}u^{\frac{1}{4}}\right) + |B_1|\right\} - p,$$

and then the assumptions $|B_1| \ge wp + 1$ and $\sum_{i=1}^n |B_i| \ge p$ yield

$$|(-b + a_1B_1) \cap E| \ge \frac{(n-3)w^3}{10^{101}}p. \tag{6}$$

For any $c \in (b - a_1B_1) \cap E$, denote by R_c the elements $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in R_{\mathcal{F}_1}(\mathcal{L})$ such that $b - a_1b_1 = c$. Take $c \in (b - a_1B_1) \cap E$. On the one hand, there is $b_1 \in B_1$ such that $c = b - a_1b_1$. On the other hand, since $E = E_{(\frac{w^{19}}{10^{1600}}p)^{n-3}}(a_2B_2, a_3B_3, \ldots, a_nB_n)$, there are at least $(\frac{w^{19}}{10^{1600}}p)^{n-2}$ vectors $(b_2, b_3, \ldots, b_n) \in B_2 \times B_3 \times \ldots \times B_n$ such that $c = a_2b_2 + a_3b_3 + \ldots + a_nb_n$. Therefore, there are at least $(\frac{w^{19}}{10^{1600}}p)^{n-3}$ vectors $(b_1, b_2, \ldots, b_n) \in B_1 \times B_2 \times \ldots \times B_n$ such that

$$b - a_1b_1 = c = a_3b_3 + a_4b_4 + \ldots + a_nb_n + a_2b_2,$$

which means that

$$|R_c| \ge (uw^3p)^{n-2} = \left(\frac{w^{19}}{10^{1600}}\right)^{n-2} p^{n-2}. \tag{7}$$

From (6) and (7), we get

$$|R_{\mathcal{F}_1}(\mathcal{L})| \ge \sum_{c\in(b-a_1B_1)\cap E} |R_c| \ge \frac{(n-3)w^3}{10^{101}}p\left(\frac{w^{19}}{10^{1600}}\right)^{n-2} p^{n-2} \ge \left(\frac{w^{19}}{10^{1600}}\right)^{n-1} p^{n-1},$$

and this concludes the proof of Theorem 3.

Acknowledgements

We would like to thank the referees for their positive and insightful comments and advices to improve this contribution.

References

[1] É. Balandraud. "Coloured solutions of equations in finite groups". In: *J. Combin. Theory Ser. A* 114.5 (2007), pp. 854–866. DOI: 10.1016/j.jcta.2006.10.002.

[2] V. Bergelson, W. A. Deuber, and N. Hindman. "Rado's theorem for finite fields". In: *Sets, graphs and numbers (Budapest, 1991)*. Vol. 60. Colloq. Math. Soc. János Bolyai. North-Holland, Amsterdam, 1992, pp. 77–88.

[3] P. Cameron, J. Cilleruelo, and O. Serra. "On monochromatic solutions of equations in groups". In: *Rev. Mat. Iberoam.* 23.1 (2007), pp. 385–395. DOI: 10.4171/RMI/499.

[4] D. Conlon. "Rainbow solutions of linear equations over $\mathbb{Z}_p$". In: *Discrete Math.* 306.17 (2006), pp. 2056–2063. DOI: 10.1016/j.disc.2006.03.070.

[5] M. Huicochea. "An inverse theorem in $\mathbb{Z}/p\mathbb{Z}$ and rainbow-free colorings". In: *Mosc. J. Comb. Number Theory* 7.2 (2017), pp. 22–58.

[6] M. Huicochea. "On the number of rainbow solutions of linear equations in $\mathbb{Z}/p\mathbb{Z}$". In: *Australas. J. Combin.* 78 (2020), pp. 118–132.

[7] M. Huicochea and A. Montejano. "The structure of rainbow-free colorings for linear equations on three variables in $\mathbb{Z}_p$". In: *Integers* 15A (2015), Paper No. A8, 19.

[8] V. Jungić, J. Nešetřil, and R. Radoičić. "Rainbow Ramsey theory". In: *Integers* 5.2 (2005), A9, 13.

[9] B. Llano and A. Montejano. "Rainbow-free colorings for $x + y = cz$ in $\mathbb{Z}_p$". In: *Discrete Math.* 312.17 (2012), pp. 2566–2573. DOI: 10.1016/j.disc.2011.09.005.

[10] A. Montejano and O. Serra. "Rainbow-free 3-colorings of abelian groups". In: *Electron. J. Combin.* 19.1 (2012), Paper 45, 20.

[11] A. Montejano and O. Serra. "Counting patterns in colored orthogonal arrays". In: *Discrete Math.* 317 (2014), pp. 44–52. DOI: 10.1016/j.disc.2013.11.002.

[12] O. Serra and L. Vena. "On the number of monochromatic solutions of integer linear systems on abelian groups". In: *European J. Combin.* 35 (2014), pp. 459–473. DOI: 10.1016/j.ejc.2013.06.034.

The weighted sum method for multi-objective optimization using Test Sets via Gröbner Bases

José Manuel Jiménez-Cobano[1†], Haydee Jiménez-Tafur[2‡], José María Ucha-Enríquez[3*,†]

[*]Departamento de Matemática Aplicada I, E. T. S. de Ingeniería Informática, Universidad de Sevilla, Av. Reina Mercedes s/n, Sevilla 41012, Spain
[†]Instituto de Matemáticas, Universidad de Sevilla, Avda. Reina Mercedes, s/n, Sevilla 41012, Spain
[‡]Departamento de Matemáticas. Universidad Pedagógica Nacional, Bogotá, Colombia.

Abstract

In this work we present how to use test sets of linear integer programming problems to apply the so called *weighted sum method* in bi-objective optimization. Although this method does not compute in general the complete set of non-dominated solutions, is one of the most widely used due to its simplicity. The interest of using test sets and Gröbner bases is that these combinatorial tools tell us exactly which weights lead to non-dominated solutions.

In addition, we show how to extend our approach to multi-objective non-linear integer programming as well.

1 Preliminaries

In real life problems we often need to optimize multiple objectives simultaneously. For example, an investor selecting a portfolio of assets wants to maximise his benefits and minimise his risk. These objectives can be conflicting, and thus some trade-offs are needed. As a result, a set of *Pareto-optimal solutions*, rather than a single solution, must be found.

A general multi-objective optimization problem can be written as

$$\begin{array}{rl} \min & f_1(\mathbf{x}), \ldots, f_r(\mathbf{x}) \\ s.t. & g_j(\mathbf{x}) \leq 0,\ j = 1, \ldots, J \\ & h_k(\mathbf{x}) = 0,\ k = 1, \ldots, K \\ & \mathbf{x} \in \mathbb{R}^d \end{array} \qquad \text{(MOILP)}$$

The space of the vectors of decision variables $\mathbf{x}$ is called the *search space*. The space formed by all the possible values of objective functions is called the *objective space*. Since in general there is no feasible point that minimises all the cost functions, we are concerned about the *efficient points*: those feasible points $\mathbf{x}^\star$ such that there is no feasible $\mathbf{x}$ with $f_i(\mathbf{x}) \leq f_i(\mathbf{x})$ with at least one strict inequality for $i = 1, \ldots, r$. If $\mathbf{x}^\star$ is an efficient point, $(f_1(\mathbf{x}^\star, \ldots, f_r(\mathbf{x}^\star))$ is a *non-dominated point* in the objective space. The set of all non-dominated points is usually called the *Pareto frontier*.

In this work we treat the bi-objective linear case (objectives and constraints are linear functions) with integer variables, although we will present how to extend our techniques to some non-linear problems. We provide an algebraic description of one of the classic methods to treat multiple-objective optimization: the *weighted sum method*. In [7] an algebraic approach based in test sets is proposed for the *ϵ-constraint method*.

[1]Email: josjimcob@alum.us.es.
[2]Email: hjimenezt@pedagogica.edu.co
[3]Email: ucha@us.es. Partially supported by MTM2016-74983-C2-1-R, MTM2016-75024-P and P12- FQM-2696.

Given a linear integer programming problem with a single objective function

$$\begin{array}{rl} \min & \mathbf{c}^t\mathbf{x} \\ \text{s.t.} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \in \mathbb{Z}^d_{\geq 0} \end{array} \tag{LIP}$$

a fundamental tool is the *test set* associated to $\mathbf{c}$ and A:

Definition 1. A test set of LIP is a set $T \subset \ker(A) \subset \mathbb{Z}^d$ (valid for any RHS $\mathbf{b}$) such that: 1) for any feasible solution $\mathbf{x}$ of LIP that is not optimal, there exists $\mathbf{t} \in T$ such that $\mathbf{x} - \mathbf{t}$ is feasible and $\mathbf{c}^t(\mathbf{x} - \mathbf{t}) < \mathbf{c}^t\mathbf{x}$, and 2) given the optimal solution $\mathbf{x}^\star$ of LIP,$\mathbf{x} - \mathbf{t}$ is not feasible for any $\mathbf{t} \in T$.

Test sets produced a natural way of solving LIP starting from a feasible point. Test sets can be computed using Gröbner bases with `4ti2` ([1]). They have been introduced in [10] to solve nonlinear integer problems, and in [3] or [6] real problems of Portfolio and Reliability Redundancy Allocation.

2 The weighted sum method with test sets

The weighted sum method (cf. [5]) combines all the multi-objective functions into a single objective MOILP into $w_1 f_1 + \cdots + w_r f_r$ with $\sum_{i=1}^r w_i = 1$ to express the preferences of the decision maker, so it is a natural idea to solve the set of the single objective problems

$$\begin{array}{rl} \min & (w_1\mathbf{c_1} + w_2\mathbf{c}_2)^t\mathbf{x} \\ \text{s.t.} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \in \mathbb{Z}^d_{\geq 0} \end{array} \tag{BOILW}$$

for $w_1 + w_2 = 1$ with test sets. It is well known that this method only produce the complete set of non-dominated solutions if the Pareto front is convex and it is not always clear how to select properly the w_i. The solutions obtained by this method are called *supported points.*

Using test sets computed with Gröbner bases it is possible to compute exactly which values of w_1, w_2 produce new potential efficient points. This is possible because the Gröbner bases behind the test sets have the following property: if the exponents with respect to an ordering $\prec_2$ of the polynomials of a given base for another ordering $\prec_1$ are the same, the bases with respect to both orderings are the same one (cf. [4]).

A brief description, given some c_1, c_2 and the matrix A of the constraints, of the procedure would be:

(1) Compute a test set of BOILW for $w_1 = 0, w_2 = (1 - w_1) = 1$. Compute the corresponding optimal point.

(2) Compute the exponents of the elements of the test set with respect to the formal combination $w_1c_1 + (1 - w_1)c_2$. Choose the minimum value $w^\star$ for which these costs become 0 (and is less than 1)

(3) Set $c_1 = w^\star c_1$ and go to (1).

The algorithm is correct because the number of different Gröbner basis fo a given ideal is finite. An additional advantage of using test sets computed with Gröbner bases is that this tool can describe exactly which w_i are necessary to be selected because they produce a different test set, so potentially a new optimal point. Nevertheless, two different test sets can lead to the same optimal point.

Example 2. Given the bi-objective assignment problem with costs

$$c_1 = (6, 4, 11, 10, 19, 18, 16, 10, 17), c_2 = (12, 12, 8, 15, 9, 1, 16, 4, 3)$$

and matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

the element $(0, 1, -1, 0, 0, 0, 0, -1, 1)$ of the test set corresponding to the cost $1 \cdot c_1 + 0 \cdot c_2$ has cost with respect to $w_1 c_1 + (1 - w_1)c_2$ equal to $1 - 8w_1$, so $w_1 = 1/8$ produce a combination of the cost that will lead to different test set, thus a potential new optimal point.

3 Application: reliability in redundancy allocation problems

In [2] a method to treat a family of three-objective redundancy allocation problem is presented. The functions to be optimised are the cost, weight and reliability of the system. We propose an alternative way to handle the case example (section 3) of three subsystems. We instead solve the bi-objective problem with respect to cost f_1 and weight f_2 and add the reliability f_R as an extra constraint (for which we ask a convenient value ρ), as in [10]. We have reordered the coefficients in f_2 to obtain more non-dominated solutions.

We consider the resulting problem of the form

$$\begin{array}{ll} \min & f_1, f_2 \\ \text{s.t.} & 1 \leq \sum_{i=1}^{3} \sum_{j=1}^{m_i} x_{ij} \leq 7 \\ & f_R(\mathbf{x}) \leq \rho \\ & x_{ij} \in \mathbb{Z}_{\geq 0}^{14} \end{array} \tag{1}$$

with

$$\begin{aligned} f_1 = \; & 4x_{11} + 6x_{12} + 7x_{13} + 8x_{14} + 9x_{15} + \\ & + 3x_{21} + 4x_{22} + 5x_{23} + 7x_{24} + \\ & + 2x_{13} + 4x_{32} + 4x_{33} + 6x_{34} + 8x_{35}, \\ f_2 = \; & 9x_{11} + 6x_{12} + 6x_{13} + 3x_{14} + 2x_{15} + \\ & + 12x_{21} + 3x_{22} + 2x_{23} + 2x_{24} + \\ & + 10x_{13} + 6x_{32} + 4x_{33} + 3x_{34} + 2x_{35} \end{aligned}$$

and $m_1 = m_3 = 5, m_2 = 4$.

Applying our method to this problem produces the consecutive values of $w_0 = 1/3, 1/2, 1/2$ and $3/4$, and the following list of 6 optimal points of the linear part:

$$\begin{gathered} (0,0,0,0,1,0,0,1,0,0,0,0,0,1), (0,0,0,0,1,0,0,1,0,0,0,0,1,0,0), \\ (0,0,0,1,0,0,1,0,0,0,0,1,0,0), (1,0,0,0,0,0,1,0,0,0,0,0,1,0,0), \\ (1,0,0,0,0,0,1,0,0,1,0,0,0,0), (1,0,0,0,0,1,0,0,0,1,0,0,0,0) \end{gathered}$$

For each of these optimal points we solve the non-linear corresponding problems with the strategy of walkback (as it is explained in [6]) for the values of ρ for which we are interested in, and obtain a suitable subset of the Pareto front of the non-linear Problem 1. For $\rho = 0.99$ the complete set of supported non-dominated points obtained is

$$\begin{gathered} \{(0,0,0,0,5,0,1,3,0,0,0,0,0,5), \\ (1,1,0,0,0,0,3,0,0,2,0,0,0,0), \\ (2,0,0,0,0,1,1,0,0,2,0,0,0,0)\} \end{gathered}$$

We consider that the approach of solving this problem for different values of ρ is more convenient that the one proposed in [2] in which 6112 non-dominated points are reported. The size of this set is unmanageable for decision maker.

4 Conclusions

We have presented an algebraic description of the weighted sum method to compute the supported non-dominated solutions of a bi-objective linear integer programming problem. A generalization to any number of objective functions is a work in progress and requires a complete understanding of how to calculate the subset of the Gröbner fan of the ideal associated with the constraints corresponding to the orders associated to the combinations of the costs.

In addition we have presented how to extend the algebraic weighted sum method to some multi-objective non-linear integer problems, specifically to a redundancy allocation problem of the literature.

References

[1] 4ti2 team. *4ti2—a software package for algebraic, geometric and combinatorial problems on linear spaces*. www.4ti2.de. 2015.

[2] D. Cao, A. E. Murat, and R. B. Chinnam. "Efficient exact optimization of multi-objective redundancy allocation problems in series-parallel systems". In: *Reliab. Eng. Syst. Saf.* 111 (2013), pp. 154–163.

[3] F. Castro, J. Gago, I. Hartillo, J. Puerto, and J. M. Ucha. "An algebraic approach to integer portfolio problems". In: *European J. Oper. Res.* 210.3 (2011), pp. 647–659. DOI: 10.1016/j.ejor.2010.11.007.

[4] D. A. Cox, J. Little, and D. O'Shea. *Using algebraic geometry*. Second. Vol. 185. Graduate Texts in Mathematics. Springer, New York, 2005, pp. xii+572.

[5] M. Ehrgott. *Multicriteria optimization*. Second Edition. Springer-Verlag, Berlin, 2005, pp. xiv+323.

[6] J. Gago, I. Hartillo, J. Puerto, and J. M. Ucha. "Exact cost minimization of a series-parallel reliable system with multiple component choices using an algebraic method". In: *Comput. Oper. Res.* 40.11 (2013), pp. 2752–2759. DOI: 10.1016/j.cor.2013.05.019.

[7] M. I. Hartillo-Hermoso, H. Jiménez-Tafur, and J. M. Ucha-Enríquez. "An exact algebraic ϵ-constraint method for bi-objective linear integer programming based on test sets". In: *European J. Oper. Res.* 282.2 (2020), pp. 453–463. DOI: 10.1016/j.ejor.2019.09.032.

[8] T. Mora and L. Robbiano. "The Gröbner fan of an ideal". In: *J. Symbolic Comput.* 6.2-3 (1988), pp. 183–208. DOI: 10.1016/S0747-7171(88)80042-7.

[9] A. Schrijver. *Theory of linear and integer programming*. Wiley-Interscience Series in Discrete Mathematics. A Wiley-Interscience Publication. John Wiley & Sons, Ltd., Chichester, 1986, pp. xii+471.

[10] S. R. Tayur, R. R. Thomas, and N. R. Natraj. "An algebraic geometry algorithm for scheduling in presence of setups and correlated demands". In: *Math. Programming* 69.3, Ser. A (1995), pp. 369–401. DOI: 10.1007/BF01585566.

[11] R. Thomas. "A geometric Buchberger algorithm for integer programming". In: *Math. Oper. Res.* 20.4 (1995), pp. 864–884. DOI: 10.1287/moor.20.4.864.

Generalized Permutahedra and Positive Flag Dressians

Michael Joswig[1*,†], Georg Loho[2‡], Dante Luber[3*], Jorge Olarte[4*]

[*]Technische Universität Berlin, Chair of Discrete Mathematics/Geometry
[†]Max-Planck Institute for Mathematics in the Sciences, Leipzig
[‡]University of Twente

The full version of this work can be found in [8] and will be published elsewhere. This research is supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - The Berlin Mathematics Research Center MATH+ (EXC-2046/1, project ID 390685689) gratefully acknowledged. M. Joswig has further been supported by "Symbolic Tools in Mathematics and their Application" (TRR 195, project-ID 286237555). D. Luber was supported by "Facets of Complexity" (GRK 2434, project-ID 385256563).

1 Introduction

A *generalized permutahedron* is any polytope P whose edge directions are differences of standard basis vectors $e_i - e_j$, where $i \neq j$. In other words, they are elements of a type A root system. When the convex hull of some subset of the vertices of P yields a generalized permutahedron, the resulting polytope is called a *subpermutahedron*. We call subdivisions of P into subpermutahedra *permutahedral subdivisions*. Permutahedral subdivisions of hypersimplices $\Delta(k, n)$ are well studied. If it is regular, such a subdivision corresponds to a valuated matroid or, equivalently, to a tropical linear space. On the other hand, non-regular matroid subdivisions can be used to construct non-realizable oriented matroids [4]. For the relevant background in tropical geometry we refer to the monographs [11] and [7].

Our main contributions are the following. In Theorem 2 we characterize the regular permutahedral subdivisions of the regular permutahedra via conditions on the 2-skeleton. This may be seen as a generalization of the description of tropical linear spaces in terms of the 3-term Plücker relations to flags of linear spaces; see also [3, 5]. Further, in Theorem 3, we show that those flags which can be realized in a totally positive manner correspond to the permutahedral subdivisions whose cells are Bruhat interval polytopes. Questions of positivity in tropical Grassmannians have been studied, e.g., in [13, 14]. One motivation comes from particle physics; cf. [10, 1] and their references.

In this note we will give a brief account of our results. Additionally, we will discuss comprehensive examples. All content presented here is expanded upon in the original version of the paper [8].

2 Preliminaries

Let $\mathrm{Sym}(n)$ denote the symmetric group on n elements. The *regular permutahedron* Π_n is the convex hull of the orbit of the point $(1, 2, \ldots, n) \in \mathbb{R}^n$ under the action of $\mathrm{Sym}(n)$ by permuting coordinates. The vertices of Π_n are identified with the elements of the symmetric group. Note that $\dim \Pi_n = n - 1$.

For our purpose here we can define a *matroid* as a subpermutahedron of a hypersimplex; cf. [7, Theorem 10.4]. In more standard terminology that would be the *base polytope* of the matroid. A *flag* $F = (F_1, \ldots, F_k)$ is a sequence of nonempty increasing subsets $F_1 \subset \cdots \subset F_k$, where F_i is referred to as the ith *constituent* of F. Our work is primarily concerned with flags where each constituent

[1]Email: joswig@math.tu-berlin.de
[2]Email: g.loho@utwente.nl
[3]Email: luber@math.tu-berlin.de
[4]Email: olarte@math.tu-berlin.de

is a subset of $[n] = \{1, 2, \ldots, n\}$. Such a flag is *full* if $k = n$. A *flag matroid* $\mathcal{M}$ now is a sequence of matroids $(M_1, \ldots, M_k)$ of consecutive rank, where the convex hull of the indicator vectors of $(M_i \times \{1\}) \cup (M_{i+1} \times \{0\})$ is a subpermutahedron of $\Delta(i+1, n+1)$ for each $i \in [k]$. As with matroids, this is not the only definition of a flag matroid, however it is known to be equivalent [3, Theorem 5.1.2]. A flag matroid $\mathcal{M}$ is identified with flags of bases $(B_1, \ldots, B_k)$, where $B_i \in \mathcal{B}(M_i)$.

Let $\mathcal{M} = (M_1, \ldots, M_k)$ be a flag matroid. The generalized permutahedron

$$\operatorname{conv}\left(\sum_{i=1}^{k} e_{B_i} | B_1 \subset \cdots \subset B_k, B_i \in \mathcal{B}(M_i)\right)$$

is the *flag matroid polytope* of $\mathcal{M}$. In this way, flags of bases are identified with vertices of the flag matroid polytope. We focus on the *full uniform flag matroid*. That is, the sequence of matroids $\mathcal{M} = (M_1, \ldots, M_n)$ where $M_d = \binom{[n]}{d}$. Observe that Π_n is the flag matroid polytope of the full uniform flag matroid on $[n]$. Indeed, Π_n is the Minkowski sum of the hypersimplices $\Delta(1, n) + \cdots + \Delta(n, n)$. It is known that every subpermutahedron of Π_n is a flag matroid polytope [2, Theorem 1.11.1].

Let $\tau_i = (i, i+1)$ denote an adjacent transposition in $\operatorname{Sym}(n)$. Recall that every element $\sigma \in \operatorname{Sym}(n)$ can be expressed as a product of transpositions. An expression $\sigma = \tau_{i_1} \ldots \tau_{i_k}$ of minimal length is a *reduced word* for σ. The *strong Bruhat order* is a partial ordering on $\operatorname{Sym}(n)$, defined by the relation $\sigma_1 \leq \sigma_2$ if some reduced word for σ_2 contains a subsequence, possibly non-consecutive, that is a reduced word for σ_1. A *Bruhat interval* is the set $[\sigma_1, \sigma_2] = \{\gamma \in \operatorname{Sym}(n) : \sigma_1 \leq \gamma \leq \sigma_2\}$.

The polytope $\operatorname{conv}([\sigma_1, \sigma_2]) \subseteq \Pi_n$, is the *Bruhat interval polytope* of $[\sigma_1, \sigma_2]$. Note that every Bruhat interval polytope is a subpermutahedron of Π_n, but the converse does not hold [9].

Example 1. Consider the quadrangle $\operatorname{conv}(123, 213, 312, 321)$, which is a subpermutahedron of the hexagon Π_3; see Figure 1 (left). This is not a Bruhat interval polytope since the inclusion of the bottom element 123 and the top element 321 would imply that it is the entire permutahedron. Yet it is a flag matroid base polytope, with constituents $(\{1, 3\}, \{12, 13, 23\}, \{123\})$.

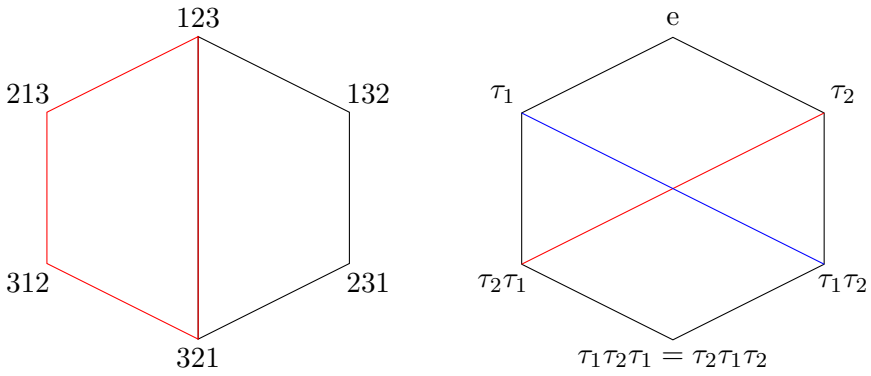


Figure 1: Left: regular hexagon Π_3 subdivided into two flag polytopes, neither of which are Bruhat interval polytopes. Right: Bruhat order of Sym(3).

Let $(\mu_1, \ldots, \mu_k)$ be a sequence of valuated matroids on a flag matroid $(M_1, \ldots, M_k)$, where the M_i are of consecutive rank d_i. If, for each $i \in [k-1]$, $S \in \binom{[n]}{d_i - 1}$ and $j, k, l \notin S$, the minimum in

$$\min(\mu_i(Sj) + \mu_{i+1}(Skl), \mu_i(Sk) + \mu_{i+1}(Sjl), \mu_i(Sl) + \mu_{i+1}(Sjk))$$

is attained at least twice, we call $(\mu_1, \ldots, \mu_k)$ a *valuated flag matroid*. The tropical polynomials above are the *3-term tropical incidence relations*. A sequence of valuated matroids is a valuated flag matroid if and only if their associated tropical linear spaces form a flag [6, Theorem 1].

Let $(\mu_1, \ldots, \mu_n)$ be a valuated uniform flag matroid. The identification of vertices of Π_n with flags of bases yields a height function $\sum_{i=1}^n \mu_i : \mathrm{Sym}(n) \to \mathbb{R}$, defined by $\sigma \mapsto \sum_{i=1}^n \mu_i(B_i)$, where $\sigma \cong (B_1, \ldots, B_n)$. From [3] we know that a regular subdivision of Π_n is permutahedral if and only if it is induced by a height function $\sum_{i=1}^n \mu_i : \mathrm{Sym}(n) \to \mathbb{R}$, where $(\mu_1, \ldots, \mu_n)$ is a valuated uniform flag matroid. Thus, regular permutahedral subdivisions of Π_n are identified with the support of a fan in $\mathbb{R}^{2^n}$ known as the *uniform flag Dressian* $\mathrm{FlDr}(n)$. The uniform flag Dressian is isomorphic to the subfan of the secondary fan of Π_n in $\mathbb{R}^{n!}$ whose cones contain height functions inducing permutahedral subdivisions. Here, we mostly make use of this embedded version of $\mathrm{FlDr}(n)$. Recall that the *Dressian* $\mathrm{Dr}(k, n)$ is the subfan of the secondary fan of $\Delta(k, n)$ which is cut out by the 3-term Plücker relations; cf. [11, Chapter 4] and [7, §10.5]. It parameterizes the valuated matroids of rank k on n elements. By [2], permutahedral subdivisions of Π_n consist of cells that are flag matroid polytopes, corresponding to full flag matroids on $[n]$.

3 Results

Our first main theorem tells us that a subdivision of Π_n is permutahedral exactly if each 2-dimensional face is subdivided into generalized permutahedra. This result also demonstrates the connection of permutahedral subdivisions to tropical geometry.

Theorem 2. *A height function $w\colon \mathrm{Sym}(n) \to \mathbb{R}$ induces a permutahedral subdivision if and only if it induces a permutahedral subdivision of the 2-skeleton of Π_n. That is:*

(HEX) for every hexagon $abcdef$ (labeled cyclically) in the 2-skeleton of Π_n, we have

- *(HXE)* $w(a) + w(c) + w(e) = w(b) + w(d) + w(f)$,
- *(HXM) the maximum in* $\max(w(a) + w(d), w(b) + w(e), w(c) + w(f))$ *is attained twice;*

(SQR) for every square face $abcd$ of Π_n (labeled cyclically): $w(a) + w(c) = w(b) + w(d)$.

Proof Sketch. We begin by duplicating the edges of the n-permutahedron, giving each two opposite orientations. We denote this doubly directed version of the vertex-edge graph of Π_n by $G^{\pm}_{\Pi_n} = (\mathrm{Sym}(n), E^{\pm})$, with the usual identification of vertices with permutations. We do the same for each hypersimplex $\Delta(d, n)$ with $d \leq n$, denoting the twice oriented 1-skeleton by $G^{\pm}_{\Delta(d,n)} = \left(V_d, E^{\pm}_d\right)$. Note that an indicator vector $e_A \in V_d$ is identified with its corresponding set in $\binom{[n]}{d}$. We make use of the following. These statements are proven in more generality in Section 4 of [8].

(A_1) A function $g : E^{\pm} \to \mathbb{R}$ that takes the same value on parallel edges of the squares in $G^{\pm}_{\Pi_n}$ induces functions $g_d : E^{\pm}_d \to \mathbb{R}$ (for each $d \leq n$) such that $g_d(e_A, e_B) = g(v, w)$, where (v, w) is an edge on the face of Π_n obtained by extending $(A \cap B, A \cup B)$ to full flags.

(A_2) A function $g : E^{\pm}_d \to \mathbb{R}$ whose values sum to 0 around the edges of every triangle in $G^{\pm}_{\Delta}(d, n)$ induces a function $f : V_d \to \mathbb{R}$ such that $g(v, w) = f(v) - f(w)$ for every edge $(v, w) \in E^{\pm}_d$.

Let $w : \mathrm{Sym}(n) \to \mathbb{R}$ be a height function satisfying (HXE), (HXM), and (SQR). Define $h' : E^{\pm} \to \mathbb{R}$ such that $h(a, b) = w(a) - w(b)$. Using (SQR) and (A_1), we obtain functions $h_d : E^{\pm}_d \to \mathbb{R}$ for each hypersimplex $\Delta(d, n)$. Then (HXE), and the identification of vertices of Π_n with flags of bases, implies that the h_d sum to 0 around the two dimensional faces of $\Delta(d, n)$. Therefore, via (A_2), we obtain $g_d : \binom{[n]}{d} \to \mathbb{R}$ such that $w(\sigma) = \sum_{d=1}^n g_d(B_d)$, where $\sigma \cong (B_1, \ldots, B_n)$. It follows from (HXM) that g_d and g_{d+1} satisfy the 3-term incidence relations for each $d \in [n-1]$. Thus $(g_1, \ldots, g_n)$ forms a full valuated flag matroid such that $w = \sum_{d=1}^n g_d$. By [3], w induces a permutahedral subdivision of Π_n. □

The above shows that the embedding of the flag Dressian, as a *set*, is cut out in $\mathbb{R}^{n!}$ by the tropical hypersurfaces and equations given in Theorem 2 by (HXE), (HXM) and (SQR). This is equivalent to being able to determine if a subdivision of Π_n is permutahedral by looking at the induced subdivision of the 2-skeleton. While this embedding of the flag Dressian still has a fan structure, it will not be a proper subfan of the secondary fan of Π_n.

The subdivisions in Figure 2 induce the same subdivision in the 2-skeleton of Π_4. However, they are distinct. In the secondary fan of Π_4 these would correspond to distinct cones. In the fan obtained via the tropical and linear conditions of Theorem 2, the height functions inducing these subdivisions are in the same cone C. In this case, C is spanned by four rays. In the secondary fan of Π_4, we would see two cones each spanned by three rays, intersecting in a common face of dimension two. Therefore the fan structure on the embedded $\mathrm{FlDr}(n)$ induced by (HXE), (HXM), and (SQR) is a coarsening fan structure in the secondary fan of Π_n. From the perspective of the 2-skeleton, this means that we can determine *if* a subdivision is permutahedral, however the subdivision itself is not determined. Note that this is in contrast with results concerning the Dressian, where the fan structure induced by the Plücker relations agrees with that of the secondary fan of the hypersimplex [12].

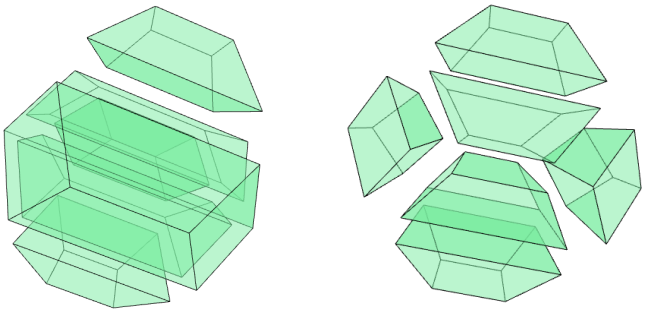


Figure 2: Two permutahedral subdivisions inducing the same pattern on the 2-skeleton.

Our second main result concerns the *positive flag Dressian*. That is, full flags of valuated matroids that satisfy the *positive 3-term Plücker relations*. This is the space of full flags of valuated matroids such that each constituent satisfies the following: For each $S \in \binom{[n]}{d-2}$ and $i < j < k < l$ in $[n] \setminus S$ we have

$$\mu(Sik) + \mu(Sjl) = \min(\mu(Sij) + \mu(Skl), \mu(Sil) + \mu(Sjk)).$$

The positive flag Dressian contains the tropicalized *positive full flag variety*. The positive full flag variety is the space of full flags of linear spaces $L_1 \subset \cdots \subset L_n$ whose Plücker coordinates are positive. When the field is the Puiseux series, "positive" just means the coefficient of the leading term is positive. We adapt the positive 3-term Plücker relations to reflect the geometry of Π_n, as well as the regular permutahedra connection to the symmetric group.

(HXM+) For every (cyclically labeled) hexagon $abcdef$ of Π_n, where b is the lowest permutation in the Bruhat order, $w(b) + w(e) = \max(w(a) + w(d), w(c) + w(f))$.

We then have the following result.

Theorem 3. *Let* $u\colon \mathrm{Sym}(n) \to \mathbb{R}$. *Then the following are equivalent:*

1. *The function* $u = \sum_{i=1}^{n} w_i$ *for a valuated flag* $(w_1, \ldots, w_n)$ *which can be realized by a totally positive flag of linear spaces.*
2. *All polytopes in the subdivision induced by* w *are Bruhat interval polytopes.*
3. *The function* u *satisfies conditions (HXE) and (SQR) from Theorem 2 as well as (HXM+).*

4 An all encompassing example

We will conclude with the following example as a coherent illustration of the main theorems. Let $\mathbb{R}\{\{t\}\}$ denote the field of Puiseux series with the valuation $\mathrm{val}\colon \mathbb{R}\{\{t\}\} \to \mathbb{Q}$ such that $\mathrm{val}(q) = k$, where k is the lowest exponent that appears in q. Thus q is positive if the coefficient of $t^{\mathrm{val}(q)}$ is positive.

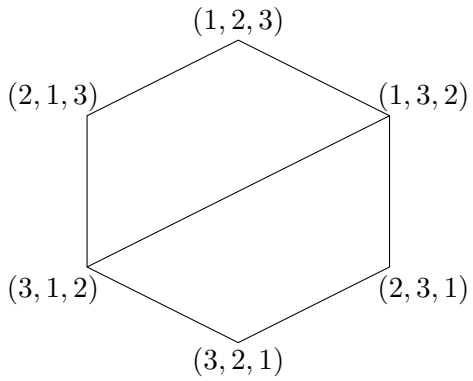


Figure 3: Subdivision of Π_3 into Bruhat interval polytopes.

We begin with the 3×3 matrix with entries from $\mathbb{R}\{\{t\}\}$ below.

$$A = \begin{bmatrix} t & 1 & t^2 \\ t^4 & 1+t^2 & t \\ 1 & 1 & 1 \end{bmatrix}$$

Let L_i be the span of the first i rows of A. Below we have the Plücker coordinates of L_1, L_2, and L_3. Note that the entries of p_{L_i} are indexed by the elements of $\binom{[3]}{i}$, ordered lexicographically. We have

$$\begin{aligned} p_{L_1} &= (t, 1, t^2) \\ p_{L_2} &= (t + t^3 - t^4, t^2 - t^6, t - t^2 - t^4) \\ p_{L_3} &= 2t - 2t^2 + t^3 - 2t^4 + t^6 \end{aligned}$$

Notice that the Plücker coordinates are all positive. That is, $L_1 \subset L_2 \subset L_3$ forms a flag of positive linear spaces. Tropicalizing the Plücker coordinates of L_1 and L_2 and L_3 yields functions $\mu_i : \binom{[3]}{i} \to \mathbb{R}$, $i \in [3]$, such that

$$\begin{aligned} \mu_1 &= (\mu_1(1), \mu_1(2), \mu_1(3)) = (1, 0, 2) \\ \mu_2 &= (\mu_2(12), \mu_2(13), \mu_2(23)) = (1, 2, 1) \\ \mu_3 &= 1 \end{aligned}$$

As the (d, n)-Dressian contains the tropical Grassmannian, we can see that $\mu_1 \in \mathrm{Dr}(1, 3)$ and $\mu_2 \in \mathrm{Dr}(2, 3)$. Recall that we can identify the vertices of Π_3 with flags of rank $(1, 2, 3)$. Define $w : \mathrm{Sym}(3) \to \mathbb{R}$ to be the function $\sum_{i=1}^{n} \mu_i$ for (μ_1, μ_2, μ_3) restricted to the vertices of Π_3. We then have

$$\begin{aligned} w((1,2,3)) = 4 \quad w((2,1,3)) = 5 \quad w((1,3,2)) = 2 \\ w((3,1,2)) = 4 \quad w((2,3,1)) = 2 \quad w((3,2,1)) = 3 \end{aligned}$$

Observe that

$$w((1,2,3)) + w((3,1,2)) + w((2,3,1)) = w((2,1,3)) + w((1,3,2)) + w((3,2,1)) = 10$$

and

$$\max\{w((1,2,3)) + w((3,2,1)), w((2,1,3)) + w((2,3,1)), w((1,3,2)) + w((3,1,2))\} = 7$$

is attained twice. That is, w meets the criteria described in Theorem 2 to induce permutahedral subdivisions on hexagons. Moreover,

$$\max\{w((1,2,3))+w((3,2,1)), w((2,1,3))+w((2,3,1)), w((1,3,2))+w((3,1,2))\}$$

attains its maximum at the $w((1,2,3))+w((3,2,1))$ entry, corresponding to the maximal and minimal elements of Sym(3) in the Bruhat order. This gives the subdivision shown in Figure 3. Indeed, we can see that the cells of this subdivision are generalized permutahedra. Further, by Theorem 3, w subdivides Π_3 into Bruhat interval polytopes (see 1).

References

[1] N. Arkani-Hamed, T. Lam, and M. Spradlin. "Positive Configuration Space". In: *Comm. Math. Phys.* 384.2 (2021), pp. 909–954. DOI: 10.1007/s00220-021-04041-x.

[2] A. V. Borovik, I. M. Gelfand, and N. White. *Coxeter matroids*. Vol. 216. Progress in Mathematics. Boston, MA: Birkhäuser, 2003, pp. xxii+264. DOI: 10.1007/978-1-4612-2066-4.

[3] M. Brandt, C. Eur, and L. Zhang. "Tropical flag varieties". In: *Adv. in Math.* 384 (2021), p. 107695. DOI: https://DOI.org/10.1016/j.aim.2021.107695.

[4] M. Celaya, G. Loho, and C. Yuen. *Oriented Matroids from Triangulations of Products of Simplices*. 2020. arXiv: 2005.01787.

[5] S. Fujishige and H. Hirai. "Compression of $M^\natural$-convex functions — Flag matroids and valuated permutohedra". In: *J. Combin. Theory Ser. A* 185 (2022), p. 105525. DOI: https://DOI.org/10.1016/j.jcta.2021.105525.

[6] M. Haque. *Tropical incidence relations, polytopes, and concordant matroids*. 2012. arXiv: 1211.2841.

[7] M. Joswig. *Essentials of tropical combinatorics*. Vol. 219. Graduate Studies in Mathematics. Providence, RI: American Mathematical Society, 2022.

[8] M. Joswig, G. Loho, D. Luber, and J. Olarte. *Generalized permutahedra and positive flag Dressians*. 2021. arXiv: 2111.13676.

[9] Y. Kodama and L. Williams. "The Full Kostant–Toda Hierarchy on the Positive Flag Variety". In: *Comm. Math. Phys.* 335.1 (2015), pp. 247–283. DOI: 10.1007/s00220-014-2203-x.

[10] T. Lukowski, M. Parisi, and L. K. Williams. *The positive tropical Grassmannian, the hypersimplex, and the $m = 2$ amplituhedron*. 2021. arXiv: 2002.06164.

[11] D. Maclagan and B. Sturmfels. *Introduction to tropical geometry*. Vol. 161. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2015, pp. xii+363.

[12] J. Olarte, M. Panizzut, and B. Schröter. "On local Dressians of matroids". In: *Algebraic and geometric combinatorics on lattice polytopes*. World Sci. Publ., Hackensack, NJ, 2019, pp. 309–329.

[13] D. Speyer and L. Williams. "The tropical totally positive Grassmannian". In: *J. Algebraic Combin.* 22.2 (2005), pp. 189–210.

[14] D. Speyer and L. Williams. "The positive Dressian equals the positive tropical Grassmannian". In: *Trans. Amer. Math. Soc. Ser. B* 8.11 (2021), pp. 330–353. DOI: 10.1090/btran/67.

Common systems of two equations over the binary field

Daniel Král'[1*], Ander Lamaison[12*], Péter Pál Pach[3†,‡]

[*]Faculty of Informatics, Masaryk University, Botanická 68A, 602 00 Brno, Czech Republic.
[†]Department of Computer Science and Information Theory, Budapest University of Technology and Economics, Műegyetem rkp. 3., H-1111 Budapest, Hungary.
[‡]MTA-BME Lendület Arithmetic Combinatorics Research Group, ELKH, Műegyetem rkp. 3., H-1111 Budapest, Hungary.

Abstract

A system of linear equations over a finite field $\mathbb{F}_q$ is said to be common if, among all two-colorings of $\mathbb{F}_q^n$, the uniform random coloring minimizes the number of monochromatic solutions asymptotically. The notion of common systems of linear equations was introduced by Saad and Wolf, as an analogue to the well-studied notion of common graphs.

Fox, Pham and Zhao characterized the common systems consisting of one equation. We study systems consisting of two equations over the binary field $\mathbb{F}_2$. We characterize, up to a finite number of cases, which systems with an odd number of variables are common. Our characterization answers a question by Kamčev, Liebenau and Morrison in the affirmative way whether there exist common systems of equations that are not translation invariant.

1 Introduction

The study of monochromatic solutions of systems of equations in colored arithmetic structures is one of the central topics in additive combinatorics. An 1892 result of this type by Hilbert [6] is often cited as the very first result in Ramsey theory, preceding Ramsey's Theorem [11] itself. Theorems by Van der Waerden [15] and Rado [10] are other famous examples of statements of this kind. However, these results only concern the existence of a monochromatic solutions, and they do not deal with the number of solutions, or with the structure of the solution set.

The problem that we study here originates from the quantitative question concerning Ramsey type results in graph theory: the notion of common graphs. This notion is closely related to one of the most well-known open problems in extremal combinatorics—Sidorenko's Conjecture. A graph H is common if, whenever the edges of K_n are colored in two colors, the number of monochromatic copies of H is asymptotically minimized for the uniformly random coloring. In other words, as n tends to infinity, the proportion of morphisms from H to K_n which are monochromatic tends to at least $2^{|E(H)|-1}$ regardless of a two-coloring of the edges of K_n. Goodman [5] proved that the triangle is common and Erdős [3] conjectured that all cliques are common. Burr and Rosta [1] extended the conjecture of Erdős and conjectured that every graph is common. Sidorenko [13] disproved the latter conjecture, by proving that a triangle with a pendant edge is not common. The conjecture of Erdős' was disproved by

[1]E-mail: dkral@fi.muni.cz. The work of D.K. and A.L. has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 648509). This publication reflects only its authors' view; the European Research Council Executive Agency is not responsible for any use that may be made of the information it contains. The authors were also supported by the MUNI Award in Science and Humanities MUNI/I/1677/2018 of the Grant Agency of Masaryk University.

[2]E-mail: lamaison@fi.muni.cz

[3]P.P.P. was supported by the Lendület program of the Hungarian Academy of Sciences (MTA) and by the National Research, Development and Innovation Office NKFIH (Grant Nr. K124171 and K129335).

Thomason [14], who proved that no clique on at least four vertices is common. More generally, any graph containing K_4 is not common [7]. We note that Sidorenko's Conjecture asserts that quasirandom graphs minimizes densities of bipartite graphs, and so if true, it would imply that all bipartite graphs are common.

We study the notion of common systems of linear equations over $\mathbb{F}_q$, which was introduced by Saad and Wolf [12]. Given a system L and a subset $A \subseteq \mathbb{F}_q^n$, we denote by $\mathrm{sol}(L; A)$ the set of elements $\mathbf{x} \in A^k$ with $L(\mathbf{x}) = 0$. The set A should be understood as one of the color classes while the other color class is $\mathbb{F}_q^n \setminus A$. We say that a system L is non-degenerate if each variable is constrained by at least one of the equations.

Definition 1. Let L be a non-degenerate, full rank system of m linear equations over k variables, on $\mathbb{F}_q$. We say that L is *c*ommon if, for any positive integer n and any $A \subseteq \mathbb{F}_q^n$ we have

$$|\mathrm{sol}(L; A)| + |\mathrm{sol}(L; \mathbb{F}_2^n \setminus A)| \geq \frac{|\mathrm{sol}(L; \mathbb{F}_2^n)|}{2^{k-1}}. \tag{1}$$

Otherwise, we say that L is uncommon.

Cameron, Cilleruelo and Serra [2] proved that every linear equation with non-zero coefficients on an odd number of variables is common. Saad and Wolf [12] proved that any equation with an even number of variables such that the coefficients can be split into pairs adding up to zero is common. The characterization of the common systems of a single linear equation was given by Fox, Pham and Zhao [4], who showed that an equation is common if and only if it belongs to one of the classes previously described. An immediate corollary of the just mentioned results is that every linear equation over the two element field $\mathbb{F}_2$ is common.

Kamčev, Liebenau and Morrison [8, 9] gave several results concerning common systems L of two or more linear equations. Some of these results give conditions on subsystems induced by L, which imply that L is uncommon; a system L induces a subsystem if the subsystem is implied by L. For example, Kamčev et al. [9] showed that if L induces a subsystem of two equations on four variables, then L is uncommon. Consequently, if every solution of L contains an arithmetic progression of length four formed by the same variables, then L is uncommon. Kamčev et al. [9] also posed several open questions. In particular, they asked whether there exists a system of linear equations of rank at least two that is common but not translation invariant.

2 Our results

Our results concern systems of two linear equations over the binary field $\mathbb{F}_2$. Since the only non-zero coefficient in $\mathbb{F}_2$ is 1, every linear system of two equations can be written in the form

$$x_1 + x_2 + \cdots + x_r = x'_1 + x'_2 + \cdots + x'_s = x''_1 + x''_2 + \cdots + x''_t$$

for some values of r, s, t. We denote this system by $L_{r,s,t}$.

In this work, we consider the case that the number $r + s + t$ of variables is odd only. While we have also obtained results for the case when the sum is even, the analysis used in the proof is more complicated and we intend to complete them and included to the journal version of this work. We discuss the differences between the even and the odd cases when we sketch the proof of our main result in Section 3. In the odd case, our main result is a characterization of common systems up to a finite number of triples (r, s, t):

Theorem 2. *(a) If all r, s and t are odd, then $L_{r,s,t}$ is common.*

(b) If r is odd, s and t are even and $t \geq 2r + s$, then $L_{r,s,t}$ is uncommon.

(c) There exists a constant C such that if r is odd, s and t are even, $s \le t < 2r + s$ and $t \ge C$, then $L_{r,s,t}$ is common.

The remaining cases can be obtained by permuting r, s and t.

The constant C in Theorem 2(c) arises from an estimate in a certain bound in the proof, and could potentially be just an artifact of the proof. In fact, numerical computations for small cases suggest that the statement holds with $C = 0$.

The significance of the condition $t \ge 2r + s$ is that it is the threshold at which the random coloring becomes locally uncommon. In Case (b), the coloring that violates (1) can be obtained by changing the color of relatively few elements of the uniformly random coloring.

The systems described in Case (c) are common, have rank two and are not translation invariant (since it contains the equation $x_1 + \cdots + x_r + x'_1 + \cdots + x'_s = 0$, with an odd number of variables, adding the same non-zero vector to all variables of a solution yields a non-solution). Hence, this answers the aforementioned question of Kamčev, Liebenau and Morrison. In fact, we can show that $L_{1,2,2}$ is common using a Cauchy-Schwarz argument.

3 Sketch of the proof

Similarly to [4], the two main techniques employed in our arguments are using a Fourier transform and replacing the original integer optimization problem by its linear relaxation.

Let $L = L_{r,s,t}$ be a system of two linear equations over the binary filed, and let $k = r + s + t$. Further, let $\mathbb{F}_2^n = R \cup B$ be a coloring of the elements of $\mathbb{F}_2^n$, i.e., the elements of one of the colors form the set R and of the other color the set B. For simplicity, we denote the variables as $\mathbf{x} = (x_1, x_2, \ldots, x_k)$. The number of monochromatic solutions can be written as

$$|\mathrm{sol}(L, R)| + |\mathrm{sol}(L, B)| = \sum_{\mathbf{x} \in \mathrm{sol}(L, \mathbb{F}_2^n)} \left(\prod_{i=1}^{k} 1_R(x_i) + \prod_{i=1}^{k} 1_B(x_i) \right) \tag{2}$$

where 1_R and 1_B are the indicator functions for the sets R and B.

A system L is common if and only if the equation (2) has a value at least $2^{n(k-2)-(k-1)}$ for any n and any function $1_R : \mathbb{F}_2^n \to \{0, 1\}$ and $1_B = 1 - 1_R$, which is equivalent to (1) with $A = R$ (if (2) drops below $2^{n(k-2)-(k-1)}$ for one n, it fails for infinitely many n's by a multiplicative factor). It can be observed that, if the system L is common, then the same inequality also holds if 1_R is replaced by any function $f : \mathbb{F}_2^n \to [0, 1]$, and 1_B by $1 - f$, i.e., it holds that

$$\sum_{\mathbf{x} \in \mathrm{sol}(L, \mathbb{F}_2^n)} \left(\prod_{i=1}^{k} f(x_i) + \prod_{i=1}^{k} (1 - f)(x_i) \right) \ge 2^{n(k-2)-(k-1)} \tag{3}$$

for all functions $f : \mathbb{F}_2^n \to [0, 1]$. The reason is as follows. If the function f is a counterexample for (3), then it is possible to obtain a partition $\mathbb{F}_2^N = R \cup B$ with $N \gg n$ by adding each $(y_1, y_2, \ldots, y_N) \in \mathbb{F}_2^N$ to R independently with probability $f(y_1, y_2, \ldots, y_n)$. With high probability, this partition will not satisfy (1).

A key step in deciding whether all functions f satisfy (3) is to consider the Fourier transform of (3). For each $y \in \mathbb{F}_2^n$, we define the Fourier coefficient

$$\hat{f}(y) = 2^{-n} \sum_{x \in \mathbb{F}_2^n} (-1)^{\langle x, y \rangle} f(x).$$

With this transformation, the condition (3) for L being common is transformed to

$$\begin{aligned} 2^{1-r-s-t} \leq & \hat{f}(0)^{r+s+t} + (1-\hat{f}(0))^{r+s+t} \\ & + \left(\hat{f}(0)^r - (\hat{f}(0)-1)^r\right) \sum_{y \in \mathbb{F}_2^n \setminus \{0\}} \hat{f}(y)^{s+t} \\ & + \left(\hat{f}(0)^s - (\hat{f}(0)-1)^s\right) \sum_{y \in \mathbb{F}_2^n \setminus \{0\}} \hat{f}(y)^{t+r} \\ & + \left(\hat{f}(0)^t - (\hat{f}(0)-1)^t\right) \sum_{y \in \mathbb{F}_2^n \setminus \{0\}} \hat{f}(y)^{r+s}, \end{aligned} \tag{4}$$

where $\hat{f}$ is the Fourier transform of any function $f : \mathbb{F}_2^n \to [0,1]$. This is the crucial point at which the parity of $r+s+t$ becomes important: the Fourier transform of the terms corresponding to the product of f contain a term, $\sum_{y \neq 0} \sum_{z \notin \{0,y\}} \hat{f}(y)^r \hat{f}(z)^s \hat{f}(y+z)^t$, which cancels out with a similar term coming from the product of $1-f$. When $r+s+t$ is even, these terms add up rather than cancel out, bringing an additional layer of complexity to the equation.

Using the transformation to (4), the proof of Theorem 2(a) is easy: the last three lines of (4) are trivially non-negative and the inequality in the first line holds by convexity of x^{r+s+t}. Theorem 2(b) is proven by giving an explicit description of a function that violates our inequality. Finally, Theorem 2(c), which is the most complex case, is established using Parseval's identity in combination with a fine case analysis.

References

[1] S. A. Burr and V. Rosta. "On the Ramsey multiplicities of graphs—problems and recent results". In: *J. Graph Theory* 4.4 (1980), pp. 347–361. DOI: 10.1002/jgt.3190040403.

[2] P. Cameron, J. Cilleruelo, and O. Serra. "On monochromatic solutions of equations in groups". In: *Rev. Mat. Iberoam.* 23.1 (2007), pp. 385–395. DOI: 10.4171/RMI/499.

[3] P. Erdős. "On the number of complete subgraphs contained in certain graphs". In: *Magyar Tud. Akad. Mat. Kutató Int. Közl.* 7 (1962), pp. 459–464.

[4] J. Fox, H. T. Pham, and Y. Zhao. "Common and Sidorenko linear equations". In: *Q. J. Math.* 72.4 (2021), pp. 1223–1234. DOI: 10.1093/qmath/haaa068.

[5] A. W. Goodman. "On sets of acquaintances and strangers at any party". In: *Amer. Math. Monthly* 66 (1959), pp. 778–783. DOI: 10.2307/2310464.

[6] D. Hilbert. "Über die Irreducibilität ganzer rationaler Functionen mit ganzzahligen Coefficienten". In: *J. Reine Angew. Math.* 110 (1892), pp. 104–129. DOI: 10.1515/crll.1892.110.104.

[7] C. Jagger, P. Štovíček, and A. Thomason. "Multiplicities of subgraphs". In: *Combinatorica* 16.1 (1996), pp. 123–141. DOI: 10.1007/BF01300130.

[8] N. Kamčev, A. Liebenau, and N. Morrison. *On uncommon systems of equations*. 2021. DOI: 10.48550/ARXIV.2106.08986.

[9] N. Kamčev, A. Liebenau, and N. Morrison. *Towards a characterisation of Sidorenko systems*. 2021. DOI: 10.48550/ARXIV.2107.14413.

[10] R. Rado. "Studien zur Kombinatorik". In: *Math. Z.* 36.1 (1933), pp. 424–470. DOI: 10.1007/BF01188632.

[11] F. P. Ramsey. "On a Problem of Formal Logic". In: *Proc. London Math. Soc. (2)* 30.4 (1929), pp. 264–286. DOI: 10.1112/plms/s2-30.1.264.

[12] A. Saad and J. Wolf. “Ramsey multiplicity of linear patterns in certain finite abelian groups”. In: *Q. J. Math.* 68.1 (2017), pp. 125–140. DOI: 10.1093/qmath/haw011.

[13] A. F. Sidorenko. “Cycles in graphs and functional inequalities”. In: *Mat. Zametki* 46.5 (1989), pp. 72–79, 104. DOI: 10.1007/BF01139620.

[14] A. Thomason. “A disproof of a conjecture of Erdős in Ramsey theory”. In: *J. London Math. Soc. (2)* 39.2 (1989), pp. 246–255. DOI: 10.1112/jlms/s2-39.2.246.

[15] B. Van Der Waerden. “Beweis einer Baudetschen Vermutung”. In: *Nieuw Archief voor Wiskunde* 15 (1927), pp. 212–216.

Linking+SensoGraph: A new graph-based method for sensory analysis

Jacob Lahne[1*], David Orden[2†], Katherine Phetxumphou[3‡], Marino Tejedor-Romero[4§]

[*]Department of Food Science & Technology, Virginia Tech, Blacksburg, VA 14060, USA.
[†]Departamento de Física y Matemáticas, Universidad de Alcalá, 28805, Spain.
[‡]Berry Tasting Lab, Driscoll's, Watsonville, CA 95076, USA.
[§]Departamento de Automática, Universidad de Alcalá, 28805, Spain.

Abstract

Sensory analysis of foods is an important task both in industry and academia. In the last few decades, several rapid methodologies have been proposed, aiming to overcome the slowness and costs of traditional trained panels. The present work introduces a new rapid method which, for the first time, uses graphs for both gathering and processing consumers' opinions. This method was tested in two sessions, smelling spice blends and tasting chocolate bars, leading to clear results comparable to those obtained by state-of-the-art methods.

1 Introduction

Identifying similarities and differences between foods is of great importance for both sensory science and industry, being particularly useful to understand how consumers perceive a product [12]. This analysis was traditionally performed by a trained panel, but the need for fine training implies large costs in time and money. Therefore, in the last few decades a number of alternative, cheaper and faster methods have been proposed [21].

Two of the most popular among these rapid methods ask the participants to somehow group the products according to their similarity. In *Sorting*, originated in psychology in 1935 [7] and first used with foods in 1995 [13], the participants are asked to distribute the products in disjoint groups according to their own criteria, without restrictions on the number of groups or the number of products in each group. In *Projective mapping*, stemming from psychology in 1964 [4] and first used with food products in 1994 [20], the participants have to position the samples on a 2D rectangular paper or screen, in such a way that similar products become positioned closer and vice versa, according to their own criteria. See Figure 1.

The data gathered by these methods can then be analyzed using different tools. For Sorting, statistical techniques such as Multidimensional Scaling (MDS) [14] or its generalization DISTATIS [1] are typically used, providing a consensus map in which similar samples become placed nearby, and vice versa. Recent works [9, 10] proposed alternative visualizations as trees. As for analyzing data from Projective Mapping, the statistical Multiple Factor Analysis (MFA) [18] is considered the standard in food science, also providing a consensus map where more similar samples are positioned closer and vice versa. The recent approach SensoGraph [15, 16] proposed an alternative graph visualization.

[1]Email: jlahne@vt.edu
[2]Email: david.orden@uah.es. Research of D. O. supported by Project PID2019-104129GB-I00 funded by MCIN/ AEI /10.13039/501100011033.
[3]Email: Kat.Phetxumphou@driscolls.com
[4]Email: marino.tejedor@uah.es. Research of M. T-R. supported by Project PID2019-104129GB-I00 funded by MCIN/ AEI /10.13039/501100011033.

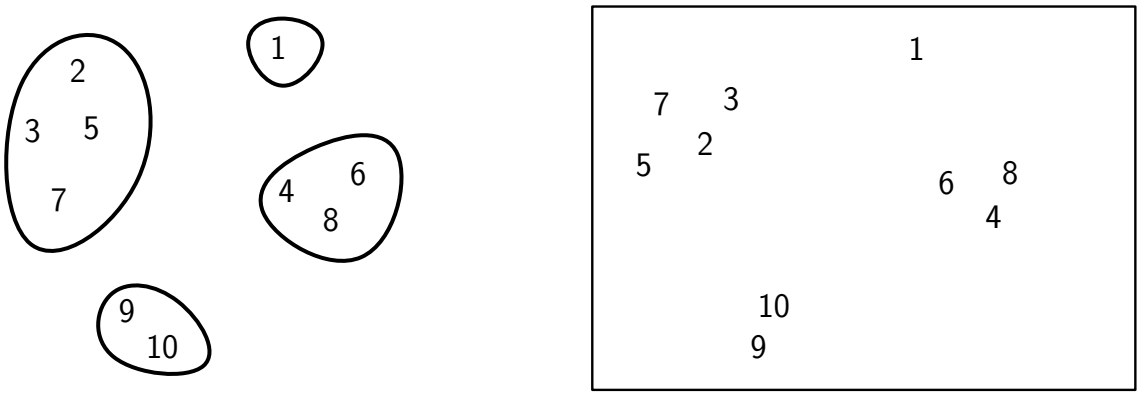


Figure 1: Examples of a participant's opinion using Sorting (left) and Projective Mapping (right).

Following this trend of using graph techniques for sensory analysis, the present work introduces a graph-based method for both gathering the consumers' opinions and processing those data. Two sessions were performed in order to test the method, whose results are discussed and compared to those obtained by state-of-the-art methods.

2 The Linking+SensoGraph graph-based method for sensory analysis

Despite their usefulness, both Sorting and Projective Mapping have disadvantages. Imagine that the example in Figure 1 asks about similarities between numbers. In Sorting (left) the groups being disjoint implies transitive similarity, i.e., A being similar to B and B being similar to C, implies A being similar to C. Thus, a participant must decide whether to group the number 2 with the other prime numbers (as in the figure) or with the other one-digit even numbers.

In Projective Mapping (right), reflecting all the similarity relations is more exhausting since, for the same groups as in the left figure, a participant should also take into account properties like the number 6 being the product of 2 and 3, hence positioning the former closer to the latter two numbers, or the number 10 being also even, hence to be positioned closer to the group of even one-digit numbers.

In order to overcome these issues, we have recently proposed the Linking method for gathering opinions [11]. In this method, the participants are asked to join with a link those pairs of samples they consider similar. For this connect-the-dots task, the samples are presented on the vertices of a regular polygon, randomizing the sample positions for each participant in order to avoid bias. See Figure 2, left.

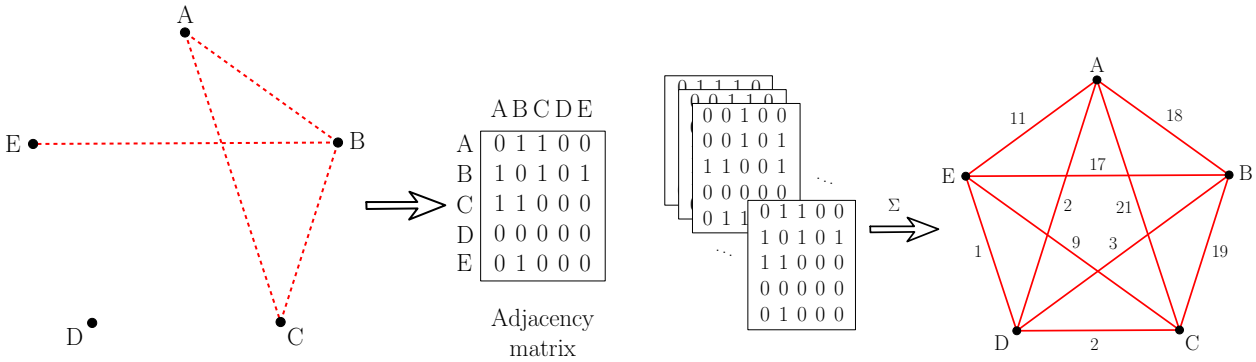


Figure 2: Left: Example of a Linking answer. Right: Adding answers gives a weighted graph.

In that work, the graph-distances between samples were then translated into a dissimilarity matrix which allowed to use the same analyzing tools as for Sorting. In the present work a different approach is explored, see Figure 2, right: Adding the 0-1 adjacency matrices of individual answers gives a *global similarity matrix* which can be interpreted as the adjacency matrix of a weighted graph. Thus, a force-directed drawing algorithm can be applied to that matrix in order to obtain a representation where more strongly connected nodes will become positioned closer, and vice versa. Graph drawing algorithms are considered an alternative to non-metric MDS in social and behavioral sciences [5], but they had not been used before for Sorting or Linking.

As in the SensoGraph method introduced in [15], we chose the standard Kamada-Kawai [8] algorithm for drawing weighted undirected graphs. This algorithm considers the edge weights of a graph as forces and lets that system of forces evolve to an equilibrium position. In this way, we obtain a consensus map where similar samples are positioned closer and vice versa. In addition, we represented larger weights as thicker and greener edges, and smaller weights as thinner and redder edges. See Figures 3 and 5.

3 Testing sessions

For the sake of comparing the results, in the two testing sessions the participants evaluated the samples using both Sorting and Linking, in a counterbalanced order. For the Sorting task, each participant received the 10 samples at the same time, in a different randomized order, and was asked to *sort into groups based on similarities* using any number of groups between two and nine, with as many samples as wanted in each group. They were informed that there was no right answer. The data were collected using the Compusense Cloud system [3] and processed using DISTATIS in the version 4.0.2. of R [19].

For the Linking task, each participant received the 10 samples at the same time, in a different randomized order, positioned as the vertices of a regular polygon, and was asked to *join with a line those pairs of products you consider similar*. The data were collected and processed using the SensoGraph system [17]. In both cases, the participants could re-taste the samples several times.

3.1 Results for a spice-blends smelling session

In a first study, a total of 58 persons (38 female and 20 male), with an average age of 29 years, performed orthonasal evaluation of aroma similarity for 10 blends of dried spices. The participants were not trained, although some of them had previously participated in other tests. All the samples were presented in foil-wrapped glass vials, in order to avoid visual discrimination. Figure 3 shows the results obtained, where the global similarity matrix (left) is arranged according to the order provided by hierarchical clustering [6].

Three groups can be observed:

- The *Cinnamon group* {Cinnamon, Cinnamon+pepper, Cinnamon+turmeric},
- the *Cardamom group* {Cardamom, Cardamom+pepper, Cardamom+turmeric}, and
- the *Pepper and Turmeric group* {Pepper, Turmeric, Pepper+turmeric}.

Together with a sample between the former two groups, Cinnamon+Cardamom, in the middle of those Cinnamon and Cardamom groups.

It is interesting to note that, out of the four basic spices used (Cardamom, Cinnamon, Pepper, and Turmeric) only Cinnamon and Cardamom dominated enough as to form their own group, which is composed by the corresponding basic spice and its blends with Pepper and Turmeric. Furthermore, Pepper and Turmeric did not form such a group, being instead grouped together and with the blend of them.

These results are comparable to those we obtained in [11] using Sorting+DISTATIS, see Figure 4, left, where more similar samples are positioned closer and confidence ellipses [2] are included.

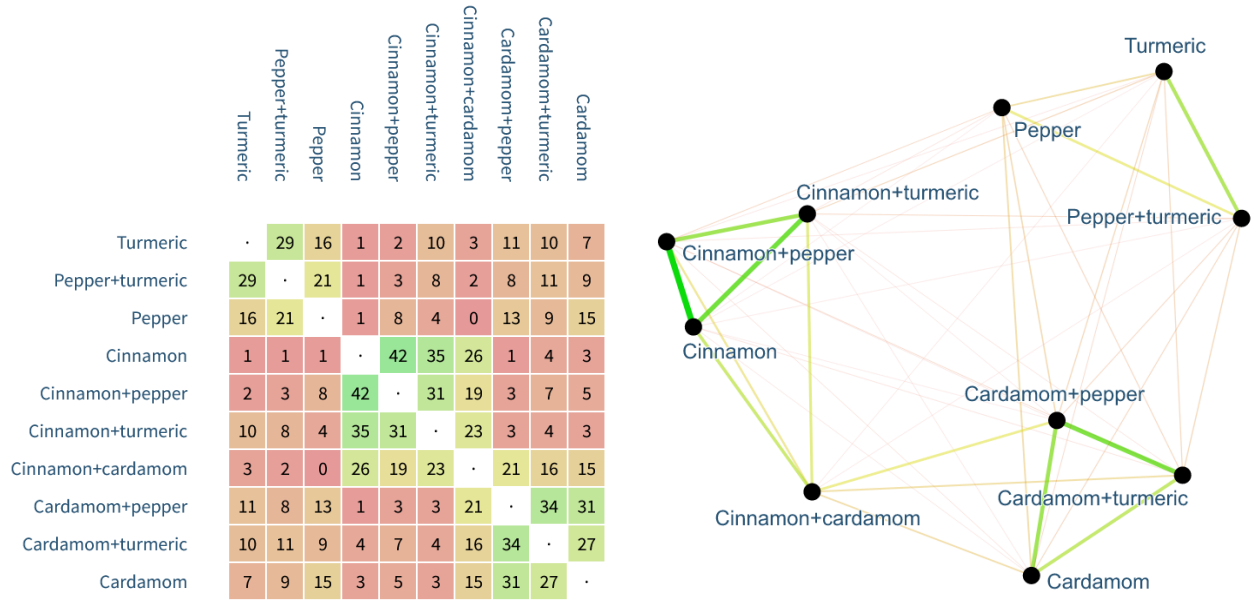


Figure 3: Results of Linking+SensoGraph for the spice-blends smelling. Left: Global similarity matrix. Right: Graph drawing by Kamada-Kawai algorithm.

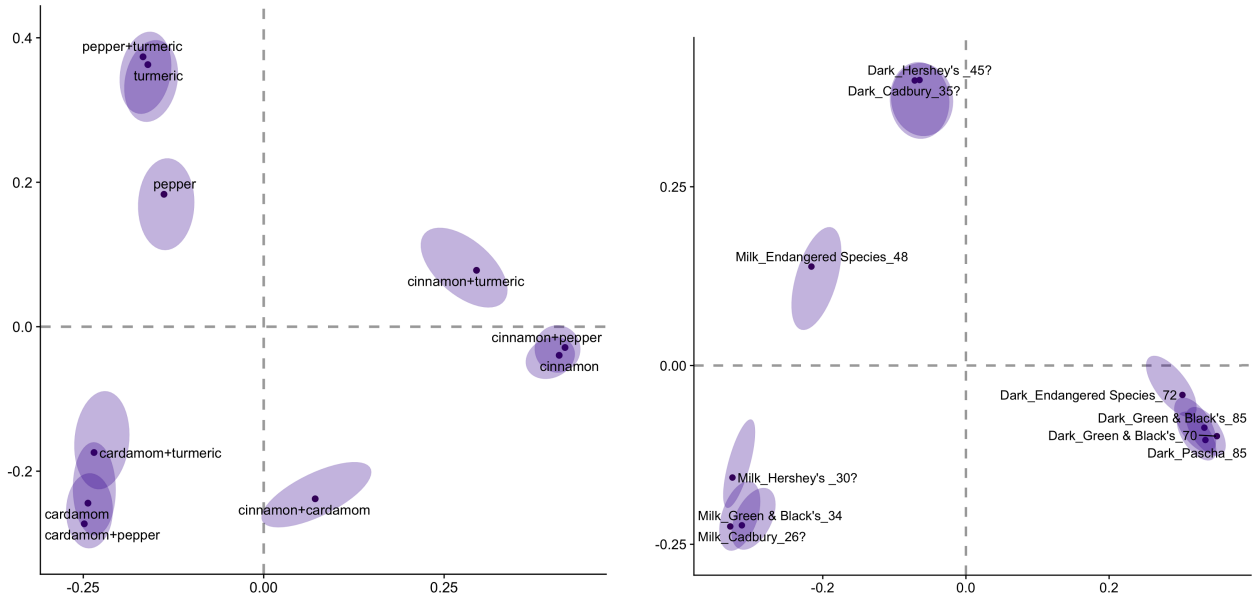


Figure 4: Left: Results for Sorting+DISTATIS for the spice-blends smelling. Right: Results for Sorting+DISTATIS for the chocolates tasting.

3.2 Results for a chocolate tasting session

In a second study, a total of 63 persons (49 female and 14 male), with an average age of 34 years, evaluated 10 commercial chocolate bars by taste and retronasal flavor. The participants were not trained, although some of them had previously participated in other tests, actually some in the previous test. All the samples were presented in souffle cups with the bars' identifying details (e.g., logos) effaced, in natural light. Figure 5 shows the results obtained, where the prefix Milk or Dark indicates the type of chocolate and the suffix indicates the percentage of cocoa content.

Again, three clear groups can be observed:

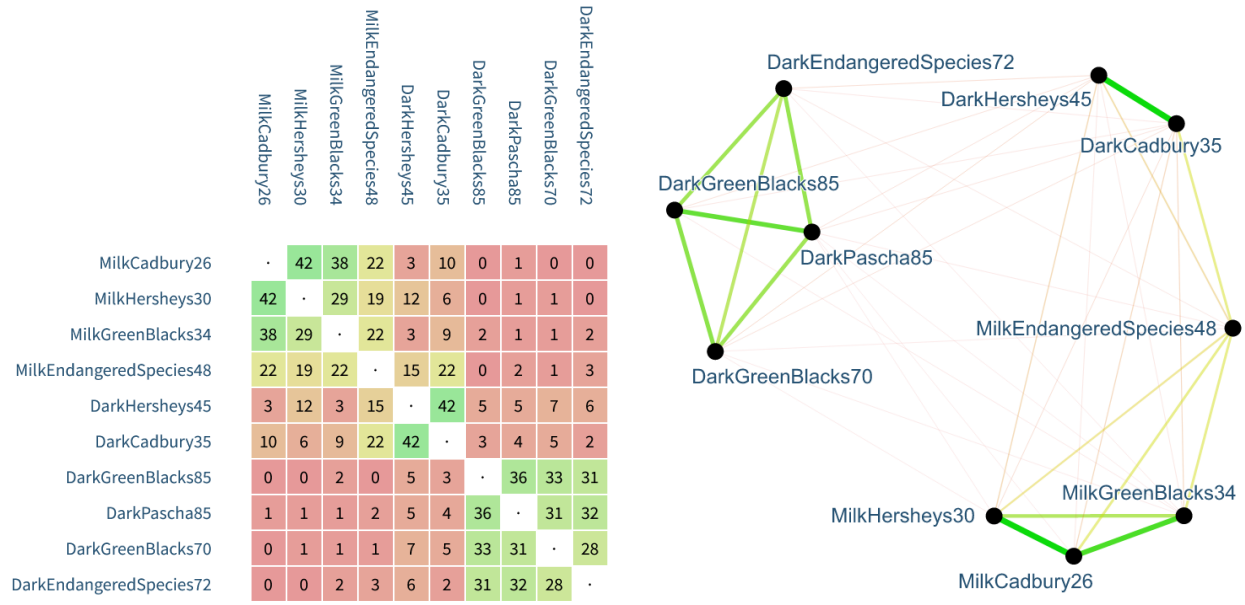


Figure 5: Results of Linking+SensoGraph for the chocolates tasting. Left: Global similarity matrix. Right: Graph drawing by Kamada-Kawai algorithm.

- The group of dark chocolates with a higher percentage of cocoa,
- the group of dark chocolates with a lower percentage of cocoa, and
- the group of milk chocolates with low percentage of cocoa.

Together with a sample between the latter two groups, MilkEndangeredSpecies48, which on one hand contains a higher percentage of cocoa than the dark chocolates in the second group and, on the other hand, is a milk chocolate like those in the third group. These results are also comparable to those we obtained in [11] using Sorting+DISTATIS, see Figure 4, right.

4 Conclusions

This work presents Linking+SensoGraph, a graph-based rapid method for assessing similarities among a set of samples. In Linking, the items to evaluate are given at the vertices of a regular polygon and the participants are asked to join with a line those they consider similar, thus creating a graph on the given nodes. The adjacency matrices of those graphs are then added to a global similarity matrix. In SensoGraph, that global similarity matrix is seen as the adjacency matrix of a weighted graph, for which the Kamada-Kawai algorithm provides a drawing where more strongly connected nodes, i.e., more similar items, become positioned closer, and vice versa.

The proposed method was tested by untrained assessors in two sessions, smelling spice blends and tasting chocolate bars. The results obtained allow to get a clear and consistent idea of the participants' opinion, are comparable to those obtained by the state-of-the-art method using Sorting+DISTATIS, and provide a new type of visualization.

References

[1] H. Abdi, D. Valentin, S. Chollet, and C. Chrea. "Analyzing assessors and products in sorting tasks: DISTATIS, theory and applications". In: *Food Quality and Preference* 18.4 (2007), pp. 627–640. DOI: https://doi.org/10.1016/j.foodqual.2006.09.003.

[2] M. Cadoret and F. Husson. "Construction and evaluation of confidence ellipses applied at sensory data". In: *Food Quality and Preference* 28.1 (2013), pp. 106–115. DOI: https://doi.org/10.1016/j.foodqual.2012.09.005.

[3] *Compusense*. https://compusense.com Accessed on 2022-03-24.

[4] C. Coombs. *A Theory of Data*. Wiley, 1964.

[5] R. Dejordy, S. P. Borgatti, C. Roussin, and D. S. Halgin. "Visualizing proximity data". In: *Field Methods* 19 (2007), pp. 239–263.

[6] W. K. Härdle and L. Simar. *Applied multivariate statistical analysis*. Springer, Heidelberg, 2015, pp. xiv+580. DOI: 10.1007/978-3-662-45171-7.

[7] W. S. Hulin and D. Katz. "The Frois-Wittmann pictures of facial expression". In: *Journal of Experimental Psychology* 18.4 (1935), pp. 482–498.

[8] T. Kamada and S. Kawai. "An algorithm for drawing general undirected graphs". In: *Information Processing Letters* 31.1 (1989), pp. 7–15. DOI: https://doi.org/10.1016/0020-0190(89)90102-6.

[9] L. Koenig, V. Cariou, R. Symoneaux, C. Coulon-Leroy, and E. Vigneau. "Additive trees for the categorization of a large number of objects, with bootstrapping strategy for stability assessment. Application to the free sorting of wine odor terms". In: *Food Quality and Preference* 89 (2021), p. 104137. DOI: https://doi.org/10.1016/j.foodqual.2020.104137.

[10] J. Lahne. "Sorting backbone analysis: A network-based method of extracting key actionable information from free-sorting task results". In: *Food Quality and Preference* 82 (2020), p. 103870. DOI: https://doi.org/10.1016/j.foodqual.2020.103870.

[11] J. Lahne, K. Phetxumphou, M. Tejedor-Romero, and D. Orden. "The free-linking task: A graph-inspired method for generating non-disjoint similarity data with food products". In: *Food Quality and Preference* 95 (2022), p. 104355. DOI: https://doi.org/10.1016/j.foodqual.2021.104355.

[12] H. T. Lawless and H. Heymann. *Sensory Evaluation of Food: Principles and Practices*. Springer Science & Business Media, 2010.

[13] H. T. Lawless, N. Sheng, and S. S. Knoops. "Multidimensional scaling of sorting data applied to cheese perception". In: *Food Quality and Preference* 6.2 (1995), pp. 91–98. DOI: https://doi.org/10.1016/0950-3293(95)98553-U.

[14] A. Mead. "Review of the Development of Multidimensional Scaling Methods". In: *Journal of the Royal Statistical Society. Series D (The Statistician)* 41.1 (1992), pp. 27–39.

[15] D. Orden, E. Fernández-Fernández, J. M. Rodríguez-Nogales, and J. Vila-Crespo. "Testing SensoGraph, a geometric approach for fast sensory evaluation". In: *Food Quality and Preference* 72 (2019), pp. 1–9. DOI: https://doi.org/10.1016/j.foodqual.2018.09.005.

[16] D. Orden, E. Fernández-Fernández, M. Tejedor-Romero, and A. Martínez-Moraian. "Geometric and statistical techniques for projective mapping of chocolate chip cookies with a large number of consumers". In: *Food Quality and Preference* 87 (2021), p. 104068. DOI: https://doi.org/10.1016/j.foodqual.2020.104068.

[17] D. Orden and M. Tejedor-Romero. *SensoGraph*. Registered at the Spanish General Register of Intellectual Property with number 16/2020/5028, Accessed on 2022-03-24, https://sensograph.it. 2020.

[18] J. Pagès. "Collection and analysis of perceived product inter-distances using multiple factor analysis: Application to the study of 10 white wines from the Loire Valley". In: *Food Quality and Preference* 16.7 (2005), pp. 642–649. DOI: https://doi.org/10.1016/j.foodqual.2005.01.006.

[19] R Core Team. *R: A Language and Environment for Statistical Computing*. http://www.R-project.org/. R Foundation for Statistical Computing. Vienna, Austria, 2007.

[20] E. Risvik, J. A. McEwan, J. S. Colwill, R. Rogers, and D. H. Lyon. "Projective mapping: A tool for sensory analysis and consumer research". In: *Food Quality and Preference* 5.4 (1994), pp. 263–269. DOI: https://doi.org/10.1016/0950-3293(94)90051-5.

[21] P. Varela and G. Ares. *Novel Techniques in Sensory Characterization and Consumer Profiling*. CRC Press, 2014.

Random projections for the distance geometry problem

Leo Liberti[1*], Benedetto Manca[2†], Pierre-Louis Poirion[3‡]

[*]LIX CNRS, École Polytechnique, Institut Polytechnique de Paris, 91128 Palaiseau, France
[†]Dip. Matematica e Informatica, Università degli Studi di Cagliari, Via Ospedale 72, 09124, Italy
[‡]RIKEN Center for Advanced Intelligence Project, Tokyo, Japan

Abstract

Random projections decrease the dimensionality of a finite set of vectors while ensuring approximate congruence, up to a multiplicative constant. Based on the theory of random projections in conic programming we derive an application of random projections to a nonconvex mathematical programming problem in distance geometry, namely that of finding the positions of the vertices of a graph in a vector space of given dimension, while ensuring that every pair of adjacent vertices is placed at a Euclidean distance equal to the corresponding edge weight.

1 Introduction

This paper is about the application of Random Projections (RP) to the Distance Geometry Problem (DGP). Insofar as RPs have been applied to Mathematical Programming (MP), this is the first time that RPs are successfully applied to a problem with nonconvex constraints.

The DGP is the following problem [9]. Given an integer $K > 0$ and a simple edge-weighted undirected graph $G = (V, E, d)$ with $d : E \to \mathbb{R}_+$, determine if there is a realization $x : V \to \mathbb{R}^K$ such that

$$\forall i < j \in V \quad \|x_i - x_j\|_2^2 = d_{ij}^2. \tag{1}$$

The DGP is useful to model inverse problems where the input is given by a subset of Euclidean distances, and the problem is to construct a consistent realization. Although the DGP is framed as a decision problem, it is defined over the real numbers, which means that it is unlikely to be in **NP** for all $K > 1$. Nonetheless, it is **NP**-hard [15].

Given $A = \{A_1, \ldots, A_n\} \in \mathbb{R}^m$, $\epsilon \in (0, 1)$, and $k = O(\epsilon^{-2} \ln(n))$, RPs are σ-sparse $k \times m$ random matrices, sampled componentwise[4] from $\mathsf{N}(0, 1/\sqrt{k\sigma})$, such that

$$\mathbf{Prob}(\ \forall i < j \leq n \quad (1-\epsilon)\|x_i - x_j\|_2 \leq \|Tx_i - Tx_j\|_2 \leq (1+\epsilon)\|x_i - x_j\|_2\ ) \geq 1 - \delta, \tag{2}$$

where $\delta = O(e^{-\mathcal{C}\phi(k)})$, $\mathcal{C}$ is a universal constant not depending on input data, and ϕ is usually linear in k. Eq. (2) is known as the Johnson-Lindenstrauss Lemma (JLL) [6], the main result in this area. It proves that RPs guarantee approximate congruence, with arbitrarily high probability (wahp) on finite point sets reduced from m to k dimensions. We note that the JLL is a high-dimensional theoretical result: if m is too small, any decent value for ϵ will already set k larger than m, trivializing the result.

[1]Email: liberti@lix.polytechnique.fr
[2]Email: bmanca@unica.it. Partly supported by grant STAGE, Fondazione Sardegna 2018.
[3]Email: pierre-louis.poirion@riken.jp
[4]Other distributions can be used, some of which discrete, e.g. [1], but in our empirical experience in projecting continuous problems what really counts is the sparsity structure of T, rather than the values of the nonzeros.

We note outright that the JLL cannot be trivially applied to the dimensionality K of the solution vectors involved in Eq. (1). Not only is K usually too small for this purpose, but the x symbols in Eq. (1) are decision variables, rather than given vectors. Given that they range over $\mathbb{R}^K$, they represent uncountable point sets: the JLL does not apply to such cases.

The key to a successful application of RPs to the DGP goes through a theory of RPs in conic programming developed in [10].

1.1 Some results about the DGP

The DGP has applications in many different dimensions. Time-synchronization network protocols motivate the DGP in one dimension $K = 1$ [16], localization of sensor networks motivate it for $K = 2$ [4], protein conformation by Nuclear Magnetic Resonance (NMR) [12] motivates it for $K = 3$. Graph embeddings in machine learning motivate it for high-dimensional values of K too [7]. For certain graph structures, one may use specific mixed-combinatorial methods [9]. In general, though, we resort to MP, a formal language for describing and solving optimization problems [8].

In this paper we make use of two simple MP formulations of the DGP: the *slack formulation*

$$\mathsf{P} \equiv \begin{cases} \min\limits_{\substack{x \in \mathbb{R}^{nK} \\ s^+ \geq 0, s^- \geq 0}} & \sum\limits_{\{i,j\} \in E} (s^+_{ij} + s^-_{ij}) \\ \forall \{i,j\} \in E & \|x_i - x_j\|_2^2 = d_{ij}^2 + s^+_{ij} - s^-_{ij}, \end{cases} \tag{3}$$

and the *quartic formulation*

$$\min_{x \in \mathbb{R}^{nK}} \sum_{\{i,j\} \in E} (\|x_i - x_j\|_2^2 - d_{ij}^2)^2. \tag{4}$$

We note that both are non-convex Nonlinear Programs (NLP), and that both are exact reformulations of one another.

1.2 Some results about RPs

There is by now an extensive body of work on RPs [5, 17], ranging from new JLL proofs, to efficient ways to sample RPs, to applications in data science, to computational testing. More recently, the field of application of RPs was generalized to optimization problems, which poses new challenges: (a) dealing with the fact that decision variables may represent infinite (rather than finite) vector sets, and (b) inferring approximate feasibility and optimality from approximate congruence. Least-squares optimization subject to black-box convex constraints was discussed in [14], Linear Programming (LP) in [19], Conic Programming (CP) in [10], Quadratic Programming (QP) in [3].

The work most relevant to the current paper is [10], which gives derived approximate feasibility and optimality results for LP, Second-Order Cone Programming (SOCP) and Semidefinite Programming (SDP) using the language of Formally Real Jordan Algebras.

2 Applying RPs to the slack formulation

We let P be the DGP formulation in Eq. (3), and T be a random projector. We define a *projected DGP formulation* $T\mathsf{P}$ similarly to [10]. For a standard-form CP

$$\mathsf{C} \equiv \min\{\langle C, X\rangle \mid \forall i \leq m \ \langle A_i, X\rangle = b_i \wedge X \succeq 0\}$$

we write the linear constraints as $[A \odot X = b] \equiv [\forall i \leq m \ \mathsf{vec}(A)_i{}^\top \mathsf{vec}(X) = b_i]$, where $\mathsf{vec}(M)$ is the vector obtained by stacking the columns of M, and define the projected CP

$$T\mathsf{C} \equiv \min\{\langle C, X\rangle \mid TA \odot X = Tb \wedge X \succeq 0\}.$$

The result of this projection is that the number of constraints goes from m down to k (see Eq. (2)), which hopefully implies that it can be solved more efficiently.

First, we note that we can limit the scope of RPs to a given subset of rows. If our linear system $A \odot X = b$ consists of two stacked subsystems $A^1 \odot X = b^1$ and $A^2 \odot X = b^2$, it suffices to define $\hat{T} = \begin{pmatrix} I & 0 \\ 0 & T \end{pmatrix}$ to obtain the projected system $\hat{T}A \odot X = \hat{T}b$ consisting of the two stacked subsystems $A^1 \odot X = b^1$ and $TA^2 \odot X = Tb^2$. This allows us to project subsets of rows of a MP formulation, and still claim validity of all of the results in [10].

Next, we note that Eq. (1) can be reformulated exactly as follows:

$$\left.\begin{array}{rcl} \forall \{i,j\} \in E \quad X_{ii} - 2X_{ij} + X_{jj} & = & d_{ij}^2 \\ X & = & xx^\top. \end{array}\right\} \tag{5}$$

This reformulation is at the basis of the SDP relaxation of the DGP, which replaces $X = xx^\top$ by $X \succeq 0$. The linear subsystem $A \odot X = b$, written as $\forall \{i,j\} \in E \; \langle A^{ij}, X \rangle = d_{ij}^2$, is defined on a set of $m = |E|$ symmetric $n \times n$ matrices A^{ij} having $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ as their $\{i,j\}$-th main 2×2 minor.

We now apply a $k \times n^2$ RP T to Eq. (5) so that only the first m rows are projected. For the $\{i,j\}$-th row of $A \odot X = b$, the application of T to $(\mathsf{vec}(A^{ij}), d_{ij}^2)$ yields a random combination of all of the rows of (A, b), hence:

$$\forall h \leq k \quad \mathsf{vec}((TA)_h) = \sum_{\{i,j\} \in E} T_{h,\{i,j\}} \mathsf{vec}(A^{ij}) = \sum_{\{i,j\} \in E} T_{h,\{i,j\}} d_{ij}^2.$$

In particular, we have

$$\forall h \leq k \quad \sum_{\{i,j\} \in E} T_{h,\{i,j\}} (X_{ii} - 2X_{ij} + X_{jj} - d_{ij}^2) = 0. \tag{6}$$

We now substitute all of the X variables back with their definition $X = xx^\top$, which yields:

$$\forall h \leq k \quad \sum_{\{i,j\} \in E} T_{h,\{i,j\}} (\|x_i - x_j\|_2^2 - d_{ij}^2) = 0. \tag{7}$$

The derivation of Eq. (7) is valid as long as the DGP instance is feasible (otherwise [10, Thm. 3.2] no longer holds, since it is based on conic duality). We address this issue heuristically, by employing Eq. (7) in the slack DGP formulation, which is always feasible. Our projected DGP formulation turns out to be:

$$T\mathsf{P} \equiv \left\{ \begin{array}{ll} \min\limits_{\substack{x \in \mathbf{R}^{nK} \\ s^+ \geq 0, s^- \geq 0}} & \sum\limits_{h \leq k} (s_h^+ + s_h^-) \\ \forall h \leq k & \sum\limits_{\{i,j\} \in E} T_{h,\{i,j\}} (\|x_i - x_j\|_2^2 - d_{ij}^2) \;=\; s_h^+ - s_h^-. \end{array} \right. \tag{8}$$

As for the error of Eq. (7) w.r.t. Eq. (5), we prove the following result.

Theorem 1. *Assume the DGP instance $G = (V, E, d)$ is feasible, and $\bar{X}$ is a solution of Eq. (6) that also satisfies $\mathsf{rk}(\bar{X}) = \bar{x}\bar{x}^\top$ for some $\bar{x} \in \mathbb{R}^{nK}$. Then*

$$\forall u > 0 \quad \mathsf{Prob}\big[\|A \odot \bar{X} - b\|_2 \leq \delta\theta^2(\mathcal{C}\sqrt{\ln n} + 2u)/\sqrt{k}\big] \geq 1 - 2e^{-u^2},$$

where $\delta = \max\limits_{v \in V} \sqrt{\deg(v)}$, θ is an upper bound to $\|x\|_1$ over every feasible realization x of the DGP, and $\mathcal{C}$ is a universal constant.

Proof. Let $C = \{A \odot X - b \mid X = xx^\top \wedge \|x\|_1 \leq \theta\}$. By [18, Ex. 9.1.8], we have that, for all $u > 0$, the following holds with probability $1 - 2e^{-u^2}$:

$$\sup_{A\odot X - b \in C} \left| \|TA \odot X - Tb\|_2 - \sqrt{k}\|A \odot X - b\|_2 \right| \leq \mathcal{C}w(C) + u\mathsf{rad}(C),$$

where $w(C)$ is the Gaussian width and $\mathsf{rad}(C) = \sup_{Y\in C} \|Y\|_2$. Since $A \odot \bar{X} - b \in C$, we can apply the above to $\bar{x}$. Since $TA \odot \bar{X} = Tb$ by definition of $\bar{X}$, we obtain: $\sqrt{k}\|A \odot X - b\|_2 \leq \mathcal{C}w(C) + u\mathsf{rad}(C)$.

We now compute estimates of $w(C)$ and $\mathsf{rad}(C)$. By the definition of the Gaussian width, we have $w(C) = \mathbb{E}_{g\sim\mathsf{N}(0,I_m)} \sup_{A\odot X - b\in C} \langle g, A \odot X - b\rangle$. Since Gaussian widths are invariant by affine translations, we obtain $w(C) = \mathbb{E}_{g\sim\mathsf{N}(0,I_m)} \sup_{A\odot X \in C+b} \langle A^\top g, \mathsf{vec}(X)\rangle$. By definition of C, we replace X by $xx^\top$ and rewrite the sup quantification, yielding $w(C) = \mathbb{E}_{g\sim\mathsf{N}(0,I_m)} \sup_{\|x\|_1\leq\theta} \langle A^\top g, \mathsf{vec}(xx^\top)\rangle$. By Hölder's inequality, $\langle A^\top g, \mathsf{vec}(xx^\top)\rangle \leq \|A^\top g\|_\infty \|\mathsf{vec}(xx^\top)\|_1$. Now, note that

$$\|\mathsf{vec}(xx^\top)\|_1 = \sum_{ij} |\sum_h x_{ih}x_{jh}| \leq \sum_{ijh} |x_{ih}|\,|x_{jh}| \leq \theta^2 \tag{9}$$

by the triangular inequality, so $w(C) = \theta^2\mathbb{E}_{g\sim\mathsf{N}(0,I_m)}\|A^\top g\|_\infty$. Next, note that the p-th component of $A^\top g$ is distributed like $\mathsf{N}(0, \|A^p\|_2)$, so by [18, Ex. 2.5.10], we obtain $w(C) = \theta^2\mathcal{C}\max_{p\leq n^2} \|A^p\|_2\sqrt{\ln n^2}$. By re-defining $\mathcal{C}$ as $\mathcal{C}\sqrt{2}$ we get $w(C) = \theta^2\mathcal{C}\max_{p\leq n^2} \|A^p\|_2\sqrt{\ln n}$. Note that, by the structure of A in the DGP, for the p-th vertex couple $(v, z) \in V \times V$, the p-th column A^p of A has at most a single -1 entry if $v \neq z$, and $\deg(v)$ 1 entries if $v = z$, so $\max_p \|A^p\|_2 = \max_{v\in V} \sqrt{\deg(v)}$.

Since the DGP instance is feasible, there exists $\hat{X}$ s.t. $A \odot \hat{X} = b$ and $\hat{X} = \hat{x}\hat{x}^\top$. We therefore have $\mathsf{rad}(C) = \sup_{A\odot X - b\in C} \|A \odot (X - \hat{X})\|_2$ (by replacement of b with $A \odot \hat{X}$), whence

$$\mathsf{rad}(C) \leq \sup_{\|x\|_1\leq\theta} \|A\,\mathsf{vec}(xx^\top)\|_2 + \|A\,\mathsf{vec}(\hat{x}\hat{x}^\top)\|_2$$

by the triangular inequalities and $X = xx^\top$. Now we have

$$\|A\,\mathsf{vec}(xx^\top)\|_2 \leq \sum_{p\leq n^2} \|A^p\|_2 |\sum_h x_{i_ph}x_{j_ph}| \leq \sup_p \|A^p\|_2\,\theta^2 = \theta^2 \max_{v\in V} \sqrt{\deg(v)}$$

by Eq. (9), where (i_p, j_p) is the p-th vertex pair in $V \times V$. □

We also note that θ can be computed from the DGP instance $G = (V, E, d)$ by assuming that the realization centroid is at the origin, and taking the worst case of a realization x on a single segment: the realization diameter is then $\sum_{\{i,j\}\in E} d_{ij}$, whence one can easily derive bounds for x and hence θ.

2.1 A new DGP solution algorithm

We note that Eq. (8) is another nonconvex NLP, albeit with fewer constraints than Eq. (3). We can therefore only solve it locally in practically acceptable times.

Instead, we turn to Barvinok's naive algorithm applied to the DGP [11]: the solution $\bar{X}$ of the SDP relaxation of Eq. (5)

$$\left.\begin{array}{rcl} \forall \{i,j\} \in E \quad X_{ii} - 2X_{ij} + X_{jj} & = & d_{ij}^2 \\ X & \succeq & 0 \end{array}\right\} \tag{10}$$

is "close" to a feasible solution of the DGP, if it exists. This closeness is more precisely defined as follows: given a random $n \times K$ matrix y with each $y_{ij} \sim \mathsf{N}(0, 1/\sqrt{K})$, the $n \times K$ matrix $\bar{x} = \sqrt{\bar{X}}y$ is close to a feasible solution of Eq. (5), in the sense that the Euclidean distance between $\bar{x}$ and each of the submanifolds of $\mathbb{R}^{nK}$ defined by $\|x_i - x_j\|_2 = d_{ij}^2$ is bounded above by $O(\sqrt{\|\bar{X}\|_2 \ln(n)})$ (wahp).

Barvinok's naive algorithm also naturally applies to the projected SDP formulation derived from Eq. (6):

$$\left.\begin{array}{rcl} \forall h \leq k \quad \sum\limits_{\{i,j\}\in E} T_{h,\{i,j\}}(X_{ii} - 2X_{ij} + X_{jj} - d_{ij}^2) & = & 0 \\ X & \succeq & 0. \end{array}\right\} \tag{11}$$

Any solution $\bar{X}$ of Eq. (11) should be close (in the sense specified above) to a feasible solution of Eq. (7), i.e. Eq. (8) with zero objective value. We therefore solve Eq. (11), obtain $\bar{X}$, then compute $\bar{x} = \sqrt{\bar{X}}y$ with $y \sim \mathsf{N}(0, 1/\sqrt{K})^{nK}$, and use it as a starting point for a local NLP solver deployed on Eq. (8): this will yield a solution $\tilde{x}$, which we can then use as a starting point for a local NLP solver deployed on the quartic formulation Eq. (4).

3 Computational assessment

We consider the application of the DGP to the problem of finding the conformation of proteins in space using NMR data, which fixes $K = 3$. Typically, atomic distances up to 5.5Å can be found using NMR experiments and chemical information. We considered three small instances for validation purposes (`lavor30_6-5`, `tiny`, `names`), then derived some realistic instances from the Protein Data Bank (PDB) as follows: we downloaded the protein realization (given with a $O(10^{-3})$ precision), we computed all atomic distances, then ignored all distances larger than 5.5Å. We compare our solution methodology, labelled M_1, with a standard method M_0 consisting in deploying a local NLP solver from a random starting point.

We used Mosek 9.3 [13] in order to solve Eq. (11) (minimizing $\mathsf{trace}(X)$ in an attempt to lead the search towards matrix solutions with low rank), and IPOpt [2] to solve Eq. (3)-(4) locally. The whole testing code, including Barvinok's naive algorithm, was implemented in Python 3.

Instance	n	m	k	mde		lde		rmsd		cpu (sec.)	
				M_0	M_1	M_0	M_1	M_0	M_1	M_0	M_1
$k/m = 0.05$											
`lavor30_6-5`	30	195	10	0.058	**0.028**	**0.560**	0.583	1.857	**1.797**	**0.10**	0.30
`tiny`	37	335	17	0.000	0.000	0.003	**0.000**	2.356	**0.000**	**0.01**	0.04
`names`	86	849	42	0.306	**0.269**	2.889	**2.156**	5.151	**4.082**	**0.23**	1.99
`1guu`	150	955	48	0.118	**0.107**	**1.104**	1.910	**8.569**	9.582	**2.85**	9.06
`1guu-1`	150	959	48	0.092	0.092	**1.096**	1.157	10.217	**7.667**	**0.47**	3.14
$k/m = 0.01$											
`2kxa`	177	2711	27	0.300	**0.281**	3.223	**3.102**	**5.196**	5.770	**1.98**	5.83
`100d`	488	5741	57	0.304	**0.286**	3.689	**3.260**	**9.345**	10.526	**5.50**	25.80
$k/m = 0.005$											
`water`	648	11939	60	0.528	**0.486**	**4.139**	**4.282**	9,423	**8.402**	**11.46**	67.03
`3al1`	678	17417	87	0.320	**0.146**	4.169	**3.312**	**6.712**	7.192	**16.03**	134.01
`1hpv`	1629	18512	93	0.429	**0.425**	3.853	**3.704**	**16.710**	17.178	**48.44**	436.52
$k/m = 0.001$											
`il2`	2084	45251	45	0.163	**0.062**	4.770	**4.220**	12.042	**3.214**	**410.67**	947.92
`1tii`	5684	69800	70	**0.434**	0.453	4.682	**4.327**	26.149	**25.680**	**549.77**	4581.21

Table 1: The test set and results.

The results we obtained are given in Table 1. We indicate the instance name, the number n of vertices, the number m of edges, the number k of constraints in the projected formulation, the mean distance error ($\mathsf{mde}(x) = \mathrm{avg}(|\,\|x_i - x_j\|_2 - d_{ij}\,| \mid \{i,j\} \in E)$), the largest distance error ($\mathsf{lde}(x) = \max(|\,\|x_i - x_j\|_2 - d_{ij}\,| \mid \{i,j\} \in E)$) and the root mean square deviation (rmsd) of the solutions given by methods M_0 (standard) and M_1 (our new method) with respect to the PDB realization, as well as the CPU times (we note, however, that protein conformation is not a time-critical

task; and that it is generally preferable to obtain better conformation slowly, than worse conformations fast).

Our results show that the application of RPs to the DGP is successful, and can scale up to large sizes. We are able to retrieve slightly better results than those obtained with the standard method. The fact that it takes longer should not come as a surprise, given that the standard method M_0, aside from the choice of starting point, is essentially the last step of our new algorithm M_1.

References

[1] D. Achlioptas. "Database-friendly random projections: Johnson-Lindenstrauss with binary coins". In: *J. Comput. System Sci.* 66.4 (2003). Special issue on PODS 2001 (Santa Barbara, CA), pp. 671–687. DOI: 10.1016/S0022-0000(03)00025-4.

[2] COIN-OR. *Introduction to IPOPT: A tutorial for downloading, installing, and using IPOPT*. 2006.

[3] C. D'Ambrosio, L. Liberti, P.-L. Poirion, and K. Vu. "Random projections for quadratic programs". In: *Math. Program.* 183.1-2, Ser. B (2020), pp. 619–647. DOI: 10.1007/s10107-020-01517-x.

[4] Y. Ding, N. Krislock, J. Qian, and H. Wolkowicz. "Sensor network localization, Euclidean distance matrix completions, and graph realization". In: *Optim. Eng.* 11.1 (2010), pp. 45–66. DOI: 10.1007/s11081-008-9072-0.

[5] P. Indyk and J. Matoušek. "Low-distortion embeddings of finite metric spaces". In: *Handbook of discrete and computational geometry (2nd ed.)* Ed. by J. E. Goodman and J. O'Rourke. Chapman & Hall/CRC, Boca Raton, FL, 2004, pp. xviii+1539. DOI: 10.1201/9781420035315.

[6] W. B. Johnson and J. Lindenstrauss. "Extensions of Lipschitz mappings into a Hilbert space". In: *Conference in modern analysis and probability (New Haven, Conn., 1982)*. Ed. by G. Hedlund. Vol. 26. Contemp. Math. Amer. Math. Soc., Providence, RI, 1984, pp. 189–206. DOI: 10.1090/conm/026/737400.

[7] L. Liberti. "A New Distance Geometry Method for Constructing Word and Sentence Vectors". In: *Companion Proceedings of the Web Conference 2020*. New York, NY, USA: Association for Computing Machinery, 2020, pp. 679–685.

[8] L. Liberti. "Distance geometry and data science". In: *TOP* 28.2 (2020), pp. 271–339. DOI: 10.1007/s11750-020-00563-0.

[9] L. Liberti, C. Lavor, N. Maculan, and A. Mucherino. "Euclidean distance geometry and applications". In: *SIAM Rev.* 56.1 (2014), pp. 3–69. DOI: 10.1137/120875909.

[10] L. Liberti, P.-L. Poirion, and K. Vu. "Random projections for conic programs". In: *Linear Algebra Appl.* 626 (2021), pp. 204–220. DOI: 10.1016/j.laa.2021.06.010.

[11] L. Liberti and K. Vu. "Barvinok's naive algorithm in distance geometry". In: *Oper. Res. Lett.* 46.5 (2018), pp. 476–481. DOI: 10.1016/j.orl.2018.06.006.

[12] T. E. Malliavin, A. Mucherino, C. Lavor, and L. Liberti. "Systematic Exploration of Protein Conformational Space Using a Distance Geometry Approach". In: *Journal of Chemical Information and Modeling* 59.10 (2019), pp. 4486–4503. DOI: 10.1021/acs.jcim.9b00215.

[13] Mosek ApS. *The `mosek` manual, Version 9*. 2019.

[14] M. Pilanci and M. J. Wainwright. "Randomized sketches of convex programs with sharp guarantees". In: *IEEE Trans. Inform. Theory* 61.9 (2015), pp. 5096–5115. DOI: 10.1109/TIT.2015.2450722.

[15] J. Saxe. "Embeddability of weighted graphs in k-space is strongly NP-hard". In: *Proceedings of 17th Allerton Conference in Communications*. Control and Computing, 1979, pp. 480–489.

[16] A. Singer. "Angular synchronization by eigenvectors and semidefinite programming". In: *Appl. Comput. Harmon. Anal.* 30.1 (2011), pp. 20–36. DOI: 10.1016/j.acha.2010.02.001.

[17] S. S. Vempala. *The random projection method*. Vol. 65. DIMACS Series in Discrete Mathematics and Theoretical Computer Science. American Mathematical Society, Providence, RI, 2004, pp. x+105. DOI: 10.1090/dimacs/065.

[18] R. Vershynin. *High-dimensional probability*. Vol. 47. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2018, pp. xiv+284. DOI: 10.1017/9781108231596.

[19] K. Vu, P.-L. Poirion, and L. Liberti. "Random projections for linear programming". In: *Math. Oper. Res.* 43.4 (2018), pp. 1051–1071. DOI: 10.1287/moor.2017.0894.

On discrete Borell-Brascamp-Lieb type inequalities for big negative parameters

Eduardo Lucas Marín[1]

Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100-Murcia, Spain

This research is part of the project PGC2018-097046-B-I00, supported by MCIN/AEI/10.13039/501100011033/ FEDER "Una manera de hacer Europa".

Abstract

In this work we derive a class of discrete Borell-Brascamp-Lieb type inequalities, connected with the α-mean of the integrals of integrable functions, when α takes big negative values.

1 Introduction

The well-known Brunn-Minkowski inequality states that for any pair of convex bodies $K, L \subset \mathbb{R}^n$ and any $\lambda \in [0,1]$, we have

$$\operatorname{vol}\left((1-\lambda)K + \lambda L\right)^{1/n} \geq \operatorname{vol}(K)^{1/n} + \operatorname{vol}(L)^{1/n}, \tag{1}$$

where $\operatorname{vol}(\cdot)$ represents the volume (Lebesgue measure) and $+$ is the Minkowski (vectorial) addition. A functional equivalent was proved by Prékopa and Leindler in the 70's, and this was soon after generalized by Borell, Brascamp and Lieb [1, 2] into a uniparametric family of functional inequalities.

Theorem A (Borell, Brascamp, Lieb)**.** *Let $0 \leq \lambda \leq 1$ and $-1/n \leq \alpha \leq \infty$. Let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be integrable functions such that*

$$h\left((1-\lambda)x + \lambda y\right) \geq \mathcal{M}_\alpha^\lambda\left(f(x), g(y)\right)$$

for all $x, y \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h(z)\,\mathrm{d}z \geq \mathcal{M}_{\frac{\alpha}{n\alpha+1}}^\lambda\left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x, \int_{\mathbb{R}^n} g(y)\,\mathrm{d}y\right).$$

In this context, $\mathcal{M}_\alpha^\lambda(a,b) = \left((1-\lambda)a^\alpha + \lambda b^\alpha\right)^{1/\alpha}$ if $ab \neq 0$ (appropriately taking the limit values if $\alpha \in \{0, \pm\infty\}$) and $\mathcal{M}_\alpha^\lambda(a,b) = 0$ if $ab = 0$. It is not hard to see that if $\alpha = \infty$ and f, g, h are the characteristic functions of K, L and $(1-\lambda)K + \lambda L$, respectively, then this result reduces to Brunn-Minkowski's inequality (1).

Recently, Iglesias, Yepes Nicolás & Zvavitch [5] obtained a sharp discrete Brunn-Minkowski type inequality for the so-called *lattice-point enumerator* G, which is defined as $\mathrm{G}(M) = |M \cap \mathbb{Z}^n|$. To do this, they first showed a discrete Borell-Brascamp-Lieb inequality, and noted the necessity to extend the function h to $h^\diamond(z) = \sup_{u \in (-1,1)^n} h(z+u)$ for all $z \in \mathbb{R}^n$.

Theorem B (Iglesias, Yepes Nicolás, Zvavitch)**.** *Let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets, $0 \leq \lambda \leq 1$ and $-1/n \leq \alpha \leq \infty$. Let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be such that*

$$h\left((1-\lambda)x + \lambda y\right) \geq \mathcal{M}_\alpha^\lambda\left(f(x), g(y)\right)$$

[1]Email: eduardo.lucas@um.es

for all $x \in K$, $y \in L$. Then

$$\sum_{z \in (M+(-1,1)^n)\cap\mathbb{Z}^n} h^\diamond(z) \geq \mathcal{M}^\lambda_{\frac{\alpha}{n\alpha+1}} \left(\sum_{x\in K\cap\mathbb{Z}^n} f(x), \sum_{y\in L\cap\mathbb{Z}^n} g(y) \right),$$

where $M = (1-\lambda)K + \lambda L$.

The reason why $\alpha \geq -1/n$ is required is merely technical: the proofs of the results above rely on Hölder's inequality to simplify (and weaken) the final expression. However, when $\alpha < -1/n$, one may continue reasoning with the stronger inequality and reach a similarly interesting result. This was done in the continuous case by Dancs and Uhrin [3].

Theorem C (Dancs, Uhrin)**.** *Let $0 \leq \lambda \leq 1$ and $-\infty \leq \alpha \leq -1/n$. Let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be integrable functions such that*

$$h\left((1-\lambda)x + \lambda y\right) \geq \mathcal{M}^\lambda_\alpha \left(f(x), g(y)\right)$$

for all $x, y \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h(z)\mathrm{d}z \geq \min\left\{ (1-\lambda)^{n+1/\alpha} \int_{\mathbb{R}^n} f(x)\mathrm{d}x,\ \lambda^{n+1/\alpha} \int_{\mathbb{R}^n} g(y)\mathrm{d}y \right\}.$$

Our aim in this work is to obtain a discrete version of the Dancs-Uhrin inequality. More precisely, we prove the following result.

Theorem 1. *Let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets, $0 \leq \lambda \leq 1$ and $-\infty \leq \alpha \leq -1/n$. Let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be such that*

$$h\left((1-\lambda)x + \lambda y\right) \geq \mathcal{M}^\lambda_\alpha \left(f(x), g(y)\right)$$

for all $x \in K$, $y \in L$. Then

$$\sum_{z \in (M+(-1,1)^n)\cap\mathbb{Z}^n} h^\diamond(z) \geq \min\left\{ (1-\lambda)^{n+1/\alpha} \sum_{x\in K\cap\mathbb{Z}^n} f(x),\ \lambda^{n+1/\alpha} \sum_{y\in L\cap\mathbb{Z}^n} g(y) \right\},$$

where $M = (1-\lambda)K + \lambda L$.

Due to the Brunn-Minkowski inequality being a key step in the proof of many further geometric inequalities (e.g. the isoperimetric inequality, the Rogers-Shephard inequality, Brunn's concavity principle or Grünbaum's inequality; see the thorough survey [4] or the excellent monograph [7] for a more in-depth presentation), the Borell-Brascamp-Lieb inequality can often play a significant role in the process of generalizing geometric results to the L_p setting. Furthermore, the consideration of arbitrary negative values for the parameter p has seen progress lately. As an example, Grünbaum's inequality was recently studied in this context in [6]. Consequently, a discrete version of Theorem C can constitute a useful tool for future discretization efforts.

2 Proof of the main result

We will use the techniques developed in [5]. Let us denote the i-th canonical unit vector by e_i, $i = 1, \ldots, n$, and the orthogonal projection onto the n-th coordinate line, $\mathrm{lin}\{e_n\}$, by $\pi_n(\cdot)$. Additionally, for any set $K \subset \mathbb{R}^n$ and any $m \in \mathbb{R}$, we will write as $K(m)$ the $(n-1)$-dimensional section of K orthogonal to e_n at height m, i.e.,

$$K(m) = \{(z_1, \ldots, z_{n-1}) \in \mathbb{R}^{n-1} : (z_1, \ldots, z_{n-1}, m) \in K\}.$$

Proof of Theorem 1. If either $\sum_{x\in K\cap\mathbb{Z}^n} f(x)$ or $\sum_{y\in L\cap\mathbb{Z}} g(y)$ are 0, then the result is trivial by definition. We may therefore assume that they are both strictly positive.

The proof is done by induction on the dimension. We begin by proving the one-dimensional case. In the proof of [5, Lemma 2.4] the following inequality, valid for any $\alpha\in\mathbb{R}$, was obtained:

$$\sum_{z\in M+(-1,1)^n\cap\mathbb{Z}} h^\diamond(z) \geq \mathcal{M}_\alpha^\lambda(a,b)\left(\frac{1-\lambda}{a}\sum_{x\in K\cap\mathbb{Z}} f(x) + \frac{\lambda}{b}\sum_{y\in L\cap\mathbb{Z}} g(y)\right), \tag{2}$$

where $a=\max_{x\in K} f(x)>0$ and $b=\max_{y\in L} g(y)>0$.

Now, denoting by $r=\sum_{x\in K\cap\mathbb{Z}} f(x)>0$, $s=\sum_{y\in L\cap\mathbb{Z}} g(y)>0$ and $t=b/a>0$, and doing a direct computation, the right-hand side of (2) rewrites into

$$(1-\lambda)r\left(1-\lambda+\lambda t^\alpha\right)^{1/\alpha} + \lambda s\left((1-\lambda)t^{-\alpha}+\lambda\right)^{1/\alpha} := \varphi(t).$$

Derivating and simplifying, we get

$$\varphi'(t) = \lambda(1-\lambda)t^{\alpha-1}\left((1-\lambda)+\lambda t^\alpha\right)^{(1-\alpha)/\alpha}\left(r-st^{-\alpha-1}\right),$$

which shows that φ is strictly increasing from 0 to $(r/s)^{-1/(\alpha+1)}$, and strictly decreasing afterwards. Therefore, the infimum is attained at either $\lim_{t\to 0^+}\varphi(t)=\lambda^{1+1/\alpha}s$ or $\lim_{t\to\infty}\varphi(t)=(1-\lambda)^{1+1/\alpha}r$, which proves the one-dimensional result.

Now, we assume the result is true for dimension $n-1$, and we prove it for dimension n. For this, we fix any $t_K\in\pi_n(K)$ and any $t_L\in\pi_n(L)$, as well as $t_\lambda=(1-\lambda)t_K+\lambda t_L$. We consider the functions $f_1,g_1,h_1:\mathbb{R}^{n-1}\longrightarrow\mathbb{R}_{\geq 0}$ given by

$$f_1(x)=f(x,t_K),\quad g_1(y)=g(y,t_L)\quad\text{and}\quad h_1(z)=h(z,t_\lambda).$$

For any $x\in K(t_K)$ and any $y\in L(t_L)$, the hypothesis implies

$$h_1\left((1-\lambda)x+\lambda y\right) \geq \mathcal{M}_\alpha^\lambda\left(f_1(x),g_1(y)\right). \tag{3}$$

Assuming first that $\alpha\leq -1/(n-1)$, we can apply the induction hypothesis to the sets $K(t_K),L(t_L)$ and the functions f_1,g_1,h_1 to obtain

$$\sum_{z\in\Omega_1\cap\mathbb{Z}^{n-1}} h_1^\diamond(z) \geq \min\left\{(1-\lambda)^{n-1+1/\alpha}\sum_{x\in K(t_K)\cap\mathbb{Z}^{n-1}} f_1(x),\ \lambda^{n-1+1/\alpha}\sum_{y\in L(t_L)\cap\mathbb{Z}^{n-1}} g_1(y)\right\}, \tag{4}$$

where $\Omega_1=(1-\lambda)K(t_K)+\lambda L(t_L)+(-1,1)^{n-1}$. It is clear that $\Omega_1\subset\Omega_2(t_\lambda)$, where $\Omega_2=M+((-1,1)^{n-1}\times\{0\})$. If we denote by $h'(z,t)=\sup_{u\in(-1,1)^{n-1}} h(z+u,t)$, $z\in\mathbb{R}^{n-1}$, then (4) yields

$$\sum_{z\in\Omega_2(t_\lambda)\cap\mathbb{Z}^{n-1}} h'(z,t_\lambda) \geq \min\left\{(1-\lambda)^{n-1+1/\alpha}\sum_{x\in K(t_K)\cap\mathbb{Z}^{n-1}} f_1(x),\ \lambda^{n-1+1/\alpha}\sum_{y\in L(t_L)\cap\mathbb{Z}^{n-1}} g_1(y)\right\}. \tag{5}$$

Now, if we consider the functions $f_2,g_2,h_2:\mathbb{R}\longrightarrow\mathbb{R}_{\geq 0}$ given by

$$f_2(t)=\sum_{x\in K(t)\cap\mathbb{Z}^{n-1}} f(x,t),\quad g_2(t)=\sum_{y\in L(t)\cap\mathbb{Z}^{n-1}} g(y,t)\quad\text{and}\quad h_2(t)=\sum_{z\in\Omega_2(t)\cap\mathbb{Z}^{n-1}} h'(z,t),$$

then (5) translates into

$$\begin{aligned} h_2\left((1-\lambda)t_K+\lambda t_L\right) &\geq \min\left\{(1-\lambda)^{n-1+1/\alpha}f_2(t_K),\lambda^{n-1+1/\alpha}g_2(t_L)\right\} \\ &= \mathcal{M}_{-\infty}^\lambda\left((1-\lambda)^{n-1+1/\alpha}f_2(t_K),\lambda^{n-1+1/\alpha}g_2(t_L)\right). \end{aligned}$$

Since this reasoning is valid for any $t_K \in \pi_n(K)$ and any $t_L \in \pi_n(L)$, we can apply the one-dimensional result, with $\alpha' = -\infty$, to the sets $\pi_n(K), \pi_n(L)$ and the functions $(1-\lambda)^{n-1+1/\alpha} f_2, \lambda^{n-1+1/\alpha} g_2, h_2$ to obtain

$$\sum_{t\in\Omega_3\cap\mathbb{Z}} h_2^\diamond(t) \geq \min\left\{(1-\lambda)^{n+1/\alpha} \sum_{t_K\in\pi_n(K)\cap\mathbb{Z}} f_2(t_K),\ \lambda^{n+1/\alpha} \sum_{t_L\in\pi_n(L)\cap\mathbb{Z}} g_2(t_L)\right\}, \tag{6}$$

where $\Omega_3 = (1-\lambda)\pi_n(K) + \lambda\pi_n(L) + (-1,1) = \pi_n\left(M + (-1,1)^n\right)$.

The proof of this case is now finished following similar steps to the one of Iglesias, Yepes Nicolás & Zvavitch. We write it here as well, for the sake of completeness. On the one hand, it is clear that

$$\sum_{t_K\in\pi_n(K)\cap\mathbb{Z}} f_2(t_K) = \sum_{x\in K\cap\mathbb{Z}^n} f(x) \quad \text{and} \quad \sum_{t_L\in\pi_n(L)\cap\mathbb{Z}} g_2(t_L) = \sum_{y\in L\cap\mathbb{Z}^n} g(y). \tag{7}$$

On the other hand, we get $\sum_{t\in\Omega_3\cap\mathbb{Z}} h_2^\diamond(t) \leq \sum_{z\in(M+(-1,1)^n)\cap\mathbb{Z}^n} h^\diamond(z)$. Indeed, since clearly

$$\Omega_2(t+w) = \left(M + \left((-1,1)^{n-1}\times\{0\}\right)\right)(t+w) \subset \left(M+(-1,1)^n\right)(t)$$

for all $w \in (-1,1)$, we have that

$$\begin{aligned}\sum_{t\in\Omega_3\cap\mathbb{Z}} h_2^\diamond(t) &= \sum_{t\in\Omega_3\cap\mathbb{Z}} \sup_{w\in(-1,1)} \sum_{x\in\Omega_2(t+w)\cap\mathbb{Z}^{n-1}} h'(x,t+w)\\ &\leq \sum_{t\in\Omega_3\cap\mathbb{Z}} \sum_{x\in(M+(-1,1)^n)(t)\cap\mathbb{Z}^{n-1}} \sup_{w\in(-1,1)} h'(x,t+w).\end{aligned}$$

A straightforward computation shows that the above expression is equal to $\sum_{z\in(M+(-1,1)^n)\cap\mathbb{Z}^n} h^\diamond(z)$. This, together with (6) and (7), finishes the proof of this case.

Now, assume that $-1/(n-1) \leq \alpha \leq -1/n$. Then, by (3), we can apply Theorem B to the sets $K(t_K), L(t_L)$ and the functions f_1, g_1, h_1 to obtain

$$\sum_{z\in\Omega_1\cap\mathbb{Z}^{n-1}} h_1^\diamond(z) \geq \mathcal{M}^\lambda_{\frac{\alpha}{(n-1)\alpha+1}}\left(\sum_{x\in K(t_K)\cap\mathbb{Z}^{n-1}} f_1(x), \sum_{y\in L(t_L)\cap\mathbb{Z}^{n-1}} g_1(y)\right).$$

Applying the same reasoning as in the previous case, and defining the same functions f_2, g_2, h_2, the above translates into

$$h_2\left((1-\lambda)t_K + \lambda t_L\right) \geq \mathcal{M}^\lambda_{\frac{\alpha}{(n-1)\alpha+1}}\left(f_2(t_K), g_2(t_L)\right)$$

for all $t_K \in \pi_n(K)$ and all $t_L \in \pi_n(L)$. Furthermore, $\alpha \leq -1/n$ implies

$$\frac{\alpha}{(n-1)\alpha+1} \leq -1,$$

and thus, we may apply the one-dimensional case with $\alpha/\left((n-1)\alpha+1\right)$ to the sets $\pi_n(K), \pi_n(L)$ and the functions f_2, g_2, h_2. Since the resulting inequality is (6), the proof of this case is concluded identically to the one of the previous one. □

References

[1] C. Borell. "Convex set functions in d-space". In: *Period. Math. Hungar.* 6.2 (1975), pp. 111–136. DOI: 10.1007/BF02018814.

[2] H. J. Brascamp and E. H. Lieb. "On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation". In: *J. Functional Analysis* 22.4 (1976), pp. 366–389. DOI: 10.1016/0022-1236(76)90004-5.

[3] S. Dancs and B. Uhrin. “On a class of integral inequalities and their measure-theoretic consequences”. In: *J. Math. Anal. Appl.* 74.2 (1980), pp. 388–400. DOI: 10.1016/0022-247X(80)90136-5.

[4] R. J. Gardner. “The Brunn-Minkowski inequality”. In: *Bull. Amer. Math. Soc. (N.S.)* 39.3 (2002), pp. 355–405. DOI: 10.1090/S0273-0979-02-00941-2.

[5] D. Iglesias, J. Yepes Nicolás, and A. Zvavitch. “Brunn-Minkowski type inequalities for the lattice point enumerator”. In: *Adv. Math.* 370 (2020), pp. 107193, 25. DOI: 10.1016/j.aim.2020.107193.

[6] F. Marín Sola and J. Yepes Nicolás. “On Grünbaum type inequalities”. In: *J. Geom. Anal.* 31.10 (2021), pp. 9981–9995. DOI: 10.1007/s12220-021-00635-y.

[7] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*. expanded. Vol. 151. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2014, pp. xxii+736.

New results for the spectra of weighted graphs of order 5

Carlos Marijuán[1], Miriam Pisonero[2]

Dept. Matemática Aplicada, University of Valladolid/IMUVA, Spain

The first author was supported in part by grant PGC2018-096446-B-C21 funded by MCIN/AEI/ 10.13039/501100011033 and by "ERDF A way of making Europe", and the second author by GIR TAMCO from UVa.

A **weighted graph** G is a triplet (V, E, w) where V is a nonempty finite set, E is a set of subsets of V with elements of cardinal 1 or 2, and $w\colon E \to \mathbb{R}^+$ is a positive real map on E. The elements of V are called **vertices**. The elements of E with cardinal two are called **edges** and the ones with cardinal one **loops**. The values of the map w are called **weights**. The **order** of a weighted graph is the number of vertices. The **adjacency matrix** of a weighted graph (V, E, w) with $V = \{v_1, \ldots, v_n\}$ is the symmetric matrix $A = (a_{ij})_{i,j=1}^n$ where $a_{ij} = a_{ji} = w(\{v_i, v_j\})$ if $i \neq j$ and $\{v_i, v_j\} \in E$, $a_{ii} = w(\{v_i\})$ if $\{v_i\} \in E$, and $a_{ij} = 0$ otherwise. The **spectrum** and the **characteristic polynomial** of a weighted graph are those of its adjacency matrix. Note that the spectrum of a weighted graph is always real.

A symmetric nonnegative matrix can be seen as the adjacency matrix of a weighted graph. In this way, symmetric nonnegative matrices and weighted graphs can be considered as equivalent objects.

For a family of real numbers $\sigma = \{\lambda_1, \ldots, \lambda_n\}$, repeats allowed, to be the spectrum of a weighted graph with adjacency matrix A, a number of necessary conditions are known. The most basic of these follow from the fact that a nonnegative matrix has real entries and nonnegative trace, and from the Perron-Frobenius theory of nonnegative matrices:

- the trace of A is nonnegative (**trace condition**): $\mathrm{Tr}(A) = \sum_{i=1}^n \lambda_i \geq 0$;
- the moments of σ of all orders are nonnegative, where the **moment of order k** of σ is the number $s_k(\sigma) = \sum_{i=1}^n \lambda_i^k = \mathrm{Tr}(A^k), \quad k \geq 1$;
  (Note that the condition $s_1 = 0$, *i.e.* trace 0, means that the weighted graphs considered have no loops and the diagonal of their adjacency matrices is null.)
- the spectral radius ρ of A is in σ (**Perron condition**); without loss of generality, this one may be taken to be λ_1: $\lambda_1 \geq |\lambda_i|, \quad i = 2, \ldots n$.

Johnson and, independently, Loewy and London obtained the first non trivial necessary conditions:

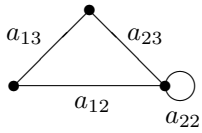
$$(s_j(\sigma))^m \leq n^{m-1} s_{jm}(\sigma), \quad j,\, m = 1, 2, \ldots.$$

The **SNIEP** (Symmetric Nonnegative Inverse Eigenvalue Problem) is the problem of characterizing all possible real spectra of weighted graphs. This problem comes from the context of nonnegative matrices.

A complete solution of the SNIEP is known only for spectra of size $n \leq 4$. For these n's the most basic necessary conditions are also sufficient: the trace and the Perron conditions characterize the problem. The table in [4] shows specific adyacency matrices and weighted graphs associated to $\sigma = \{\lambda_1 \geq \cdots \geq \lambda_n\}$ for $n \leq 4$. We would like to mention that this table contains an error for $n = 3$ with $\lambda_1 \geq 0 > \lambda_2$. Instead of what appears there, the correct adjacency matrix and the corresponding weighted graph are:

[1]Email: cmarijuan@uva.es
[2]Email: mpisonero@uva.es

$$\begin{pmatrix} 0 & \sqrt{-(\lambda_1+\lambda_2)\lambda_3} & \sqrt{\frac{\lambda_1\lambda_2\lambda_3}{\lambda_1+\lambda_2-\lambda_3}} \\ \sqrt{-(\lambda_1+\lambda_2)\lambda_3} & \lambda_1+\lambda_2+\lambda_3 & \sqrt{\frac{-(\lambda_1+\lambda_2)\lambda_1\lambda_2}{\lambda_1+\lambda_2-\lambda_3}} \\ \sqrt{\frac{\lambda_1\lambda_2\lambda_3}{\lambda_1+\lambda_2-\lambda_3}} & \sqrt{\frac{-(\lambda_1+\lambda_2)\lambda_1\lambda_2}{\lambda_1+\lambda_2-\lambda_3}} & 0 \end{pmatrix}$$



Spectra of size 5 for weighted graphs are not characterized and this problem has proven a very challenging one. Note that when we consider graphs (unweighted) their spectra are completely characterized if one is able to calculate all the spectra of the symmetric matrices with 0's and 1's of a fixed size.

In what follows we focus our attention in the SNIEP for $n = 5$. For this n there are two cases where the SNIEP is characterized. The first known result characterizes the weighted graphs without loops:

Theorem 1. (Spector [5], 2011) *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ and $s_k(\sigma) = \sum_{i=1}^5 \lambda_i^k$. Suppose $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5 \geq -\lambda_1$ and $s_1(\sigma) = 0$. Then σ is the spectrum of a weighted graph without loops if and only if the following conditions hold:*

1. $\lambda_2 + \lambda_5 \leq 0$, and *2. $s_3(\sigma) \geq 0$.*

The other known result for order 5 characterizes the weighted graphs for which the trace is bigger than or equal to half of their spectral radius:

Theorem 2. (Loewy-Spector [3], 2017) *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5$ and $\sum_{i=1}^5 \lambda_i \geq \frac{1}{2}\lambda_1$. Then, σ is the spectrum of a weighted graph if and only if the following conditions hold:*

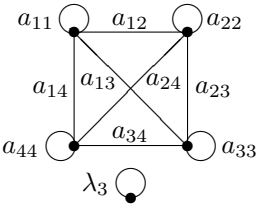
1. $\lambda_1 = \max_{\lambda\in\sigma} |\lambda|$, *2. $\lambda_2 + \lambda_5 \leq \sum_{i=1}^5 \lambda_i$, and* *3. $\lambda_3 \leq \sum_{i=1}^5 \lambda_i$.*

Many authors have studied the SNIEP for $n = 5$, see references in [4]. All of the spectra unresolved have three positive eigenvalues and two negative eigenvalues. So, as a conclusion, the spectra to consider are the ones satisfying the inequalities:

$$\lambda_1 > \lambda_2 \geq \lambda_3 > 0 > \lambda_4 \geq \lambda_5, \qquad \lambda_1 + \lambda_5 \geq 0 \text{ (Perron condition)},$$
$$\sum_{i=1}^5 \lambda_i > 0 \text{ (trace condition)} \qquad \text{and} \qquad \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 < 0.$$

The last inequality may be assumed, as otherwise $\{\lambda_1, \lambda_2, \lambda_4, \lambda_5\}$ and $\{\lambda_3\}$ would be the spectra of a weighted graph. For example:

$$(a_{ij}) = \left(\begin{array}{cccc|c} \frac{\lambda_1+\lambda_2+\lambda_4+\lambda_5}{4} & \frac{\lambda_1+\lambda_2-(\lambda_4+\lambda_5)}{4} & \frac{\lambda_1-(\lambda_2+\lambda_4)+\lambda_5}{4} & \frac{\lambda_1+\lambda_4-(\lambda_2+\lambda_5)}{4} & 0 \\ \frac{\lambda_1+\lambda_2-(\lambda_4+\lambda_5)}{4} & \frac{\lambda_1+\lambda_2+\lambda_4+\lambda_5}{4} & \frac{\lambda_1+\lambda_4-(\lambda_2+\lambda_5)}{4} & \frac{\lambda_1-(\lambda_2+\lambda_4)+\lambda_5}{4} & 0 \\ \frac{\lambda_1-(\lambda_2+\lambda_4)+\lambda_5}{4} & \frac{\lambda_1+\lambda_4-(\lambda_2+\lambda_5)}{4} & \frac{\lambda_1+\lambda_2+\lambda_4+\lambda_5}{4} & \frac{\lambda_1+\lambda_2-(\lambda_4+\lambda_5)}{4} & 0 \\ \frac{\lambda_1+\lambda_4-(\lambda_2+\lambda_5)}{4} & \frac{\lambda_1-(\lambda_2+\lambda_4)+\lambda_5}{4} & \frac{\lambda_1+\lambda_2-(\lambda_4+\lambda_5)}{4} & \frac{\lambda_1+\lambda_2+\lambda_4+\lambda_5}{4} & 0 \\ \hline 0 & 0 & 0 & 0 & \lambda_3 \end{array}\right)$$



Note that $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ is the spectrum of a weighted graph if and only if $\{K\lambda_1, \ldots, K\lambda_n\}$, for every $K > 0$, is the spectrum of a weighted graph. It is common to study spectra with Perron eigenvalue 1.

In [1], we studied spectra with single spectral radius, the other two positive eigenvalues equal and the two negative eigenvalues also equal. After normalization, the spectra studied are of the form $\{1, a, a, -(a+d), -(a+d)\}$. We used a method [1], based upon the eigenvalue interlacing inequalities for symmetric matrices, to rule out many unresolved spectra with 3 positive eigenvalues. In particular, we obtained:

Theorem 3. *Let $0 < a, d$ satisfy $a + d, 2d < 1 < a + 2d$. If $2(a+d)^3 > 1 + a^3 + (a+2d-1)^3$, then $\{1, a, a, -(a+d), -(a+d)\}$ is not the spectrum of a weighted graph of order 5.*

The method used to prove the previous result has been refined in order to be able to apply it to real spectra of size 5 with just one repeated eigenvalue, see [2].

Theorem 4. *Let $a, d_1 > 0$, $d_2 > d_1$ satisfy $a + d_2, d_1 + d_2 < 1 < a + d_1 + d_2$. If $(a + d_1)^3 + (a + d_2)^3 > 1 + a^3 + (a + d_1 + d_2 - 1)^3$, then $\sigma = \{1, a, a, -(a + d_1), -(a + d_2)\}$ is not the spectrum of a weighted graph of order 5.*

Example 5. Let us take $a = 1/3$ in Theorem 4. The hypotheses of the theorem correspond, in the d_1d_2-space, to the interior of the polygon with vertices $(0, 2/3), (1/3, 1/3), (1/2, 1/2)$ and $(1/3, 2/3)$, see the grey region on Figure 1. The inequality of the theorem is described graphically by the region above the dashed curve on Figure 1, so for those (d_1, d_2), σ is not the spectrum of a weighted graph. For example, $d_1 = 1/3$ and $d_2 = 3/5$ gives $\{1, 1/3, 1/3, -2/3, -14/15\}$ which is not the spectrum of a weighted graph. Neither $\{K, K/3, K/3, -2K/3, -14K/15\}$ for any $K > 0$, in particular $\{15, 5, 5, -10, -14\}$.

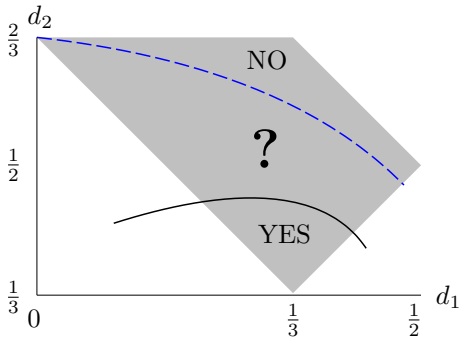


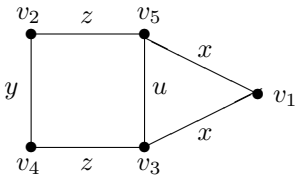
Figure 1: d_1d_2-space for $\sigma = \{1, 1/3, 1/3, -(1/3 + d_1), -(1/3 + d_2)\}$

Theorem 1 says when the translated spectrum

$$\sigma_t = \left\{1 - \tfrac{1-d_1-d_2}{5}, \tfrac{1}{3} - \tfrac{1-d_1-d_2}{5}, \tfrac{1}{3} - \tfrac{1-d_1-d_2}{5}, -\left(\tfrac{1}{3} + d_1\right) - \tfrac{1-d_1-d_2}{5}, -\left(\tfrac{1}{3} + d_2\right) - \tfrac{1-d_1-d_2}{5}\right\}$$

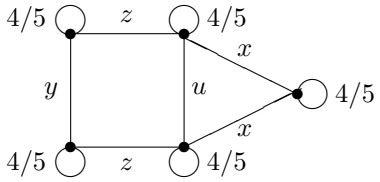
is the spectrum of a weighted graph, that is, if and only if $s_3(\sigma_t) \geq 0$ (other conditions are easily seen to be met). But σ_t is the spectrum of a weighted graph without loops if and only if σ is the spectrum of a weighted graph with constant weight on all its loops. The inequality $s_3(\sigma_t) \geq 0$ is described graphically by the region under or on the solid curve on Figure 1, so for those (d_1, d_2), σ is the spectrum of a weighted graph. For example, $d_1 = 1/3$ and $d_2 = 2/5$ gives $\{1, 1/3, 1/3, -2/3, -11/15\}$ which is the spectrum of a weighted graph. Also $\{K, K/3, K/3, -2K/3, -11K/15\}$ for every $K > 0$, in particular $\{15, 5, 5, -10, -11\}$. The techniques in [5] realize $\{15 - 4/5, 5 - 4/5, 5 - 4/5, -10 - 4/5, -11 - 4/5\}$ with

$$A = \begin{pmatrix} 0 & 0 & x & 0 & x \\ 0 & 0 & 0 & y & z \\ x & 0 & 0 & z & u \\ 0 & y & z & 0 & 0 \\ x & z & u & 0 & 0 \end{pmatrix} \quad \text{where} \quad \begin{cases} x = \frac{\sqrt{2}\sqrt{2083}}{10} \\ y = \frac{87969}{14015} \\ z = \frac{12\sqrt{646339729}}{14015} \\ u = \frac{906}{2803} \end{cases}$$



Which for $\{15, 5, 5, -10, -11\}$ gives

$$A + \tfrac{4}{5}I = \begin{pmatrix} \frac{4}{5} & 0 & x & 0 & x \\ 0 & \frac{4}{5} & 0 & y & z \\ x & 0 & \frac{4}{5} & z & u \\ 0 & y & z & \frac{4}{5} & 0 \\ x & z & u & 0 & \frac{4}{5} \end{pmatrix}$$



It is not known if the spectra corresponding to the region marked with a question mark on Figure 1, graphically the region above the solid curve and under or on the dashed curve, are spectra of weighted graphs or not. Note that Theorem 2 in not applicable because $d_1 + d_2 > 1/2$ on the whole polygon on Figure 1.

Example 6. Let us take $a = 2/3$ in Theorem 4. The hypotheses of the theorem correspond, in the d_1d_2-space, to the interior of the triangle with vertices $(0, 1/3), (1/6, 1/6)$ and $(1/3, 1/3)$, see the grey region on Figure 2. The inequality of the theorem graphically is described by the region above the dashed curve on Figure 2, so for those (d_1, d_2), σ is not the spectrum of a weighted graph. For example, $d_1 = 1/4$ and $d_2 = 5/18$ gives $\{1, 2/3, 2/3, -11/12, -17/18\}$ which is not the spectrum of a weighted graph. Neither $\{K, 2K/3, 2K/3, -11K/12, -17K/18\}$ for any $K > 0$, in particular $\{36, 24, 24, -33, -34\}$.

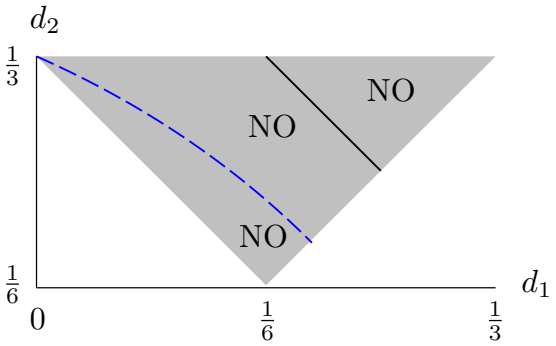


Figure 2: d_1d_2-space for $\sigma = \{1, 2/3, 2/3, -(2/3 + d_1), -(2/3 + d_2)\}$

Theorem 2 says when σ is the spectrum of a weighted graph if

$$s_1(\sigma) = 1 - d_1 - d_2 \geq \frac{1}{2} \iff d_1 + d_2 \leq \frac{1}{2}.$$

This inequality graphically is described by the region under or on the solid line on Figure 2. But these spectra clearly satisfy conditions 1 and 2 but not 3. Therefore, no spectrum corresponding to points under or on the line $d_1 + d_2 = 1/2$ is the spectrum of a weighted graph. For example, $d_1 = 1/6$ and $d_2 = 2/9$ gives $\{1, 2/3, 2/3, -5/6, -8/9\}$ which is not the spectrum of a weighted graph. Neither $\{K, 2K/3, 2K/3, -5K/6, -8K/9\}$ for every $K > 0$, in particular $\{18, 12, 12, -15, -16\}$.

Note then that no spectrum corresponding to (d_1, d_2) on the interior of the triangle on Figure 2 is the spectrum of a weighted graph. Also note that the technique of translated σ and using Theorem 1 is not applicable in this case.

Theorem 7. *Let $d, r > 0$, $a > r$ satisfy $a+d, r+2d < 1 < a+2d$. If $2(a+d)^3 > 1+a^3+(a+2d-1)^3$, then $\sigma = \{1, a, a - r, -(a + d), -(a + d)\}$ is not the spectrum of a weighted graph of order 5.*

Example 8. Let us take $a = 2/5$ in Theorem 7. The hypotheses of the theorem correspond, in the dr-space, to the interior of the triangle with vertices $(3/10, 0), (1/2, 0)$ and $(3/10, 2/5)$, see the grey region on Figure 3. The inequality of the theorem graphically is described by the region on the right hand side of the dashed curve on Figure 3, so for those (d, r), σ is not the spectrum of a weighted graph. For example, $d = 11/25$ and $r = 1/25$ gives $\{1, 2/5, 9/25, -21/25, -21/25\}$ which is not the spectrum of a weighted graph. Neither $\{K, 2K/5, 9K/25, -21K/25, -21K/25\}$ for any $K > 0$, in particular $\{25, 10, 9, -21, -21\}$.

Theorem 1 says when the translated spectrum

$\sigma_t = \left\{1 - \frac{1-r-2d}{5}, \frac{2}{5} - \frac{1-r-2d}{5}, \frac{2}{5} - r - \frac{1-r-2d}{5}, -\left(\frac{2}{5} + d\right) - \frac{1-r-2d}{5}, -\left(\frac{2}{5} + d\right) - \frac{1-r-2d}{5}\right\}$

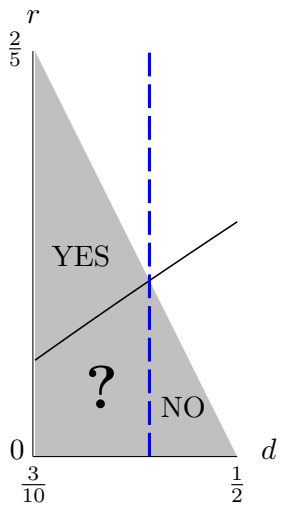
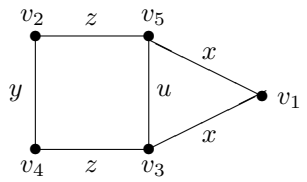


Figure 3: dr-space for $\sigma = \{1, 2/5, 2/5 - r, -(2/5 + d), -(2/5 + d)\}$

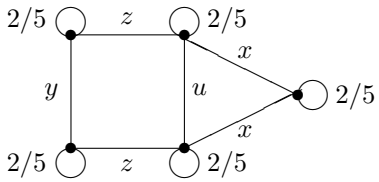
is the spectrum of a weighted graph, that is, if and only if $s_3(\sigma_t) \geq 0$ (other conditions are easily seen to be met). But σ_t is the spectrum of a weighted graph without loops if and only if σ is the spectrum of a weighted graph with constant weight on all its loops. The inequality $s_3(\sigma_t) \geq 0$ is graphically described by the region above or on the solid curve on Figure 3, so for those (d, r), σ is the spectrum of a weighted graph. For example, $d = 1/3$ and $r = 1/5$ gives $\{1, 2/5, 1/5, -11/15, -11/15\}$ which is the spectrum of a weighted graph. Also $\{K, 2K/5, K/5, -11K/15, -11K/15\}$ for every $K > 0$, in particular $\{15, 6, 3, -11, -11\}$. The techniques in [5] realize $\{15 - 2/5, 6 - 2/5, 3 - 2/5, -11 - 2/5, -11 - 2/5\}$ with

$$A = \begin{pmatrix} 0 & 0 & x & 0 & x \\ 0 & 0 & 0 & y & z \\ x & 0 & 0 & z & u \\ 0 & y & z & 0 & 0 \\ x & z & u & 0 & 0 \end{pmatrix} \qquad \text{where} \qquad \begin{cases} x = \frac{\sqrt{2}\sqrt{2357}}{10} \\ y = \frac{54093}{11785} \\ z = \frac{8\sqrt{1189}\sqrt{126654}}{11785} \\ u = \frac{2852}{2357} \end{cases}$$



Which for $\{15, 6, 3, -11, -11\}$ gives

$$A + \tfrac{2}{5}I = \begin{pmatrix} \frac{2}{5} & 0 & x & 0 & x \\ 0 & \frac{2}{5} & 0 & y & z \\ x & 0 & \frac{2}{5} & z & u \\ 0 & y & z & \frac{2}{5} & 0 \\ x & z & u & 0 & \frac{2}{5} \end{pmatrix}$$



It is not known if the spectra corresponding to the region marked with a question mark on Figure 3, graphically the region under the solid curve and on the left hand side or on the dashed curve, are spectra of weighted graphs or not. Note that Theorem 2 in not applicable because $r + 2d > 1/2$ on the whole triangle on Figure 3.

Example 9. Let us take $a = 2/3$ in Theorem 7. The hypotheses of the theorem correspond, in the dr-space, to the interior of the polygon with vertices $(1/6, 0), (1/3, 0), (1/3, 1/3)$ and $(1/6, 2/3)$, see the grey region on Figure 4. The inequality of the theorem graphically is described by the region on the right hand side of the dashed curve on Figure 4, so for those (d, r), σ is not the spectrum of a weighted graph.

Theorem 1 says when the translated spectrum

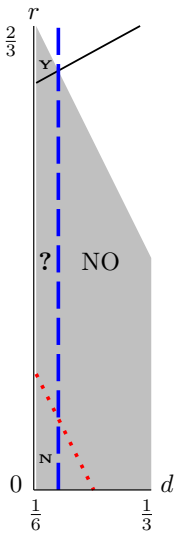


Figure 4: dr-space for $\sigma = \{1, 2/3, 2/3 - r, -(2/3 + d), -(2/3 + d)\}$

$$\sigma_t = \left\{1 - \tfrac{1-r-2d}{5}, \tfrac{2}{3} - \tfrac{1-r-2d}{5}, \tfrac{2}{3} - r - \tfrac{1-r-2d}{5}, -\left(\tfrac{2}{3} + d\right) - \tfrac{1-r-2d}{5}, -\left(\tfrac{2}{3} + d\right) - \tfrac{1-r-2d}{5}\right\}$$

is the spectrum of a weighted graph, that is, if and only if $s_3(\sigma_t) \geq 0$ (other conditions are easily seen to be met). But σ_t is the spectrum of a weighted graph without loops if and only if σ is the spectrum of a weighted graph with constant weight on all its loops. The inequality $s_3(\sigma_t) \geq 0$ is graphically described by the region above or on the solid curve on Figure 4, so for those (d, r), σ is the spectrum of a weighted graph. Theorem 2 says when σ is the spectrum of a weighted graph if

$$s_1(\sigma) = 1 - r - 2d \geq \frac{1}{2} \iff r + 2d \leq \frac{1}{2}.$$

This inequality graphically is described by the region under or on the dotted line on Figure 2. But these spectra clearly satisfy conditions 1 and 2 but not 3. Therefore, no spectrum corresponding to points under or on the line $r + 2d = 1/2$ is the spectrum of a weighted graph. It is not known if the spectra corresponding to the region marked with a question mark on Figure 4, graphically the region among the solid curve, the dotted line and the dashed curve (including only the dashed curve), are spectra of weighted graphs or not.

References

[1] C. R. Johnson, C. Marijuán, and M. Pisonero. "Ruling out certain 5-spectra for the symmetric nonnegative inverse eigenvalue problem". In: *Linear Algebra Appl.* 512 (2017), pp. 129–135. DOI: 10.1016/j.laa.2016.09.022.

[2] C. R. Johnson, C. Marijuán, and M. Pisonero. "Ruling out certain 5-spectra with one repeated eigenvalue for the symmetric NIEP". In: *Linear Algebra Appl.* 612 (2021), pp. 75–98. DOI: 10.1016/j.laa.2020.12.004.

[3] R. Loewy and O. Spector. "A necessary condition for the spectrum of nonnegative symmetric 5×5 matrices". In: *Linear Algebra Appl.* 528 (2017), pp. 206–272. DOI: 10.1016/j.laa.2016.08.006.

[4] C. Marijuán and M. Pisonero. "On spectra of weighted graphs of order ≤ 5". In: *Electronic Notes in Discrete Mathematics* 68 (2018). Discrete Mathematics Days 2018, pp. 95–100. DOI: https://doi.org/10.1016/j.endm.2018.06.017.

[5] O. Spector. “A characterization of trace zero symmetric nonnegative 5×5 matrices”. In: *Linear Algebra Appl.* 434.4 (2011), pp. 1000–1017. DOI: 10.1016/j.laa.2010.10.006.

Local parking procedures on the integers

Philippe Nadeau[1]

Univ Lyon, CNRS, Université Claude Bernard Lyon 1, UMR 5208, Institut Camille Jordan, 43 Blvd. du 11 Novembre 1918, F-69622 Villeurbanne Cedex, France

The full version of this work will be published elsewhere. This research is partially supported by the project ANR19-CE48-011-01 (COMBINÉ)

Abstract

We introduce a large class of parking procedures on $\mathbb{Z}$ generalizing the classical one. This class is characterized by natural local constraints that the procedures must satisfy. We uncover some nice combinatorics attached to such procedures, including a certain universal enumeration formula.

1 Introduction

Classical parking functions (or words) are one of the fundamental objects of enumerative and algebraic combinatorics, connected to various structures such as noncrossing partitions, hyperplane arrangements, and many others: see for instance the survey [8] and references therein. The corresponding parking procedure on $\mathbb{Z}$ was originally defined as a very simple hashing procedure [4].

We recall their definition informally: r cars want to park on an empty street where the spots are labeled by $1, 2, \dots, r$ from left to right. The cars arrive successively, and the ith car has a preferred spot v_i. If its spot is available, it parks there, and if not it parks in the nearest available spot on the right. The function $v : i \mapsto v_i$ is called a *parking function* if at the end, all cars managed to park.

The main result is that the number of parking functions is given by the simple formula $(r+1)^{r-1}$. Another standard result is the following characterization: $v : i \mapsto v_i$ is a parking function if and only if for any $k = 1, \dots, r$, there are at least k indices i such that $1 \leq v_i \leq k$.

In ongoing joint work with Vasu Tewari [5], the authors defined procedures that could be rephrased as certain parking[2] algorithms themselves. This gave the idea to define a general mathematical framework for these procedures, which is the content of this abstract.

We first describe the local procedures $\mathcal{P}$ that we want to consider. We will then see that the enumeration $(r+1)^{r-1}$ is in a sense universal for our parking procedures, see Corollary 9. We also describe some natural connection with the combinatorics of binary trees via a natural encoding, and finally define certain extensions of our model.

2 Our setup

We first slightly extend the sequence of desired spots from the cars. Let S be any set, and consider the alphabet $A = \mathbb{Z} \times S$. The set S represents some extra information: in terms of cars, one might consider its brand or color. We let $\mathrm{val} : A = \mathbb{Z} \times S \to \mathbb{Z}$ be the projection to the first factor, which represents the desired spot. As we will see the second factor in S will be carried along nicely in our construction, and we will focus mostly on the values.

[1]Email: nadeau@math.univ-lyon1.fr.

[2]We use parking in a loose fashion in this whole abstract, as we do not focus on any practical application of the procedures but rather on the combinatorial structures that they give rise to.

Remark 1. *When S is a singleton $S = \{\bullet\}$, we identify $A \cong \mathbb{Z}$ and* val *is the identity.*

A sequence of incoming cars is given by a word $a_1 a_2 \dots a_r$ with $a_i \in A = \mathbb{Z} \times S$, read from left to right. The ith car wants to park in $\mathrm{val}(a_i)$.

We want to describe certain functions

$$\mathcal{P} : A^* \to \mathrm{Fin}(\mathbb{Z}) = \{I \subset \mathbb{Z} \mid \#I < +\infty\}.$$

$\mathcal{P}(a_1 \cdots a_r)$ is the subset of occupied spots after cars with preferences $a_1, \dots, a_r \in \mathbb{Z} \times S$ have arrived successively. We define our parking procedures on $\mathbb{Z}$ –instead of a finite or semiinfinite interval– in order to get rid of any boundary effect. For an element of $\mathrm{Fin}(\mathbb{Z})$, its *(maximal) intervals* are its connected components as an induced graph of $\mathbb{Z}$.

Remark 2. *More generally, one can require $\mathcal{P}$ to be a* partial *function, that is, to be defined on a subset $L \subset A^*$. It is reasonable to require that L be closed under deleting a letter at any position, that is, that L be closed under taking subwords. For example, one can forbid certain letters in A to occur more than a certain number of times.*

In our models *all cars are able to park, and cars do not move again once they're parked*: $\mathcal{P}(\epsilon) = \emptyset$, and for any $W \in A^*, a \in A$ we have $\mathcal{P}(Wa) = \mathcal{P}(W) \sqcup \{i\}$ for a certain $i \in \mathbb{Z}$.

We denote $\mathrm{ls}(Wa) := i$, which is the spot where the last car parks. If $W = a_1 \cdots a_r$, then the function $\pi^W : \{1, \dots, r\} \to \mathcal{P}(W), i \mapsto \mathrm{ls}(a_1 a_2 \cdots a_i)$ is a bijection.

We now define the first two requirements for $\mathcal{P}$, which can be summarized as: If your desired spot is available, park there, otherwise park to the nearest spot either to the left or to the right.

- **(Lucky parking)** If $\mathrm{val}(a) \notin \mathcal{P}(W)$, then $\mathrm{ls}(Wa) = \mathrm{val}(a)$;
- **(Local move)** If $\mathrm{val}(a) \in \mathcal{P}(W)$, let $[t, u]$ be the maximal interval of $\mathcal{P}(W)$ such that $\mathrm{val}(a) \in [t, u]$. Then $\mathrm{ls}(Wa) \in \{t - 1, u + 1\}$.

To define $\mathcal{P}$, it thus suffices to determine a "rule" that picks either $\mathrm{ls}(Wa) = t-1$, *left*, or $\mathrm{ls}(Wa) = u+1$, *right*, whenever $\mathrm{val}(a) \in \mathcal{P}(W)$, that is when the desired spot is occupied. We will require such a rule to be local in some specific way described by the last two conditions.

Remark 3. *Note that a recent variation of parking functions, k-Naples functions [1], do not enter our framework. The rule there is to back up k spots before finding the first available spot on the right, which sometimes contradicts the previous requirements.*

We now come to the remaining two constraints on the functions $\mathcal{P}$. Informally, we require invariance under translation (shift), and that left/right decisions must depend only on the subsequence of cars that parked on the encountered interval.

Mathematically speaking, let $\tau : i \mapsto i + 1$ be the shift on $\mathbb{Z}$. It extends to subsets of $\mathbb{Z}$. Also, if $a = (i, s)$, define $\tau(a) = (i + 1, s)$ and extend to words.

- **(Shift invariance)** For any W, $\mathcal{P}(\tau(W)) = \tau(\mathcal{P}(W))$.
- **(Local decision)** Consider an interval I that is maximal in $\mathcal{P}(W)$ and $\mathcal{P}(W')$ for two words W and W', such that in both words the subword corresponding to the cars parked in I is identical. Then for any a such that $\mathrm{val}(a) \in I$, $\mathrm{ls}(Wa) = \mathrm{ls}(W'a)$.

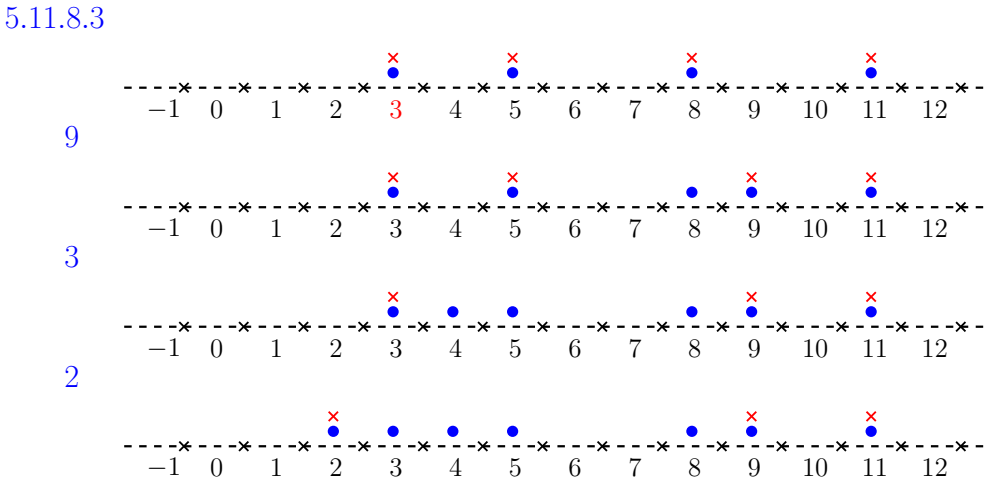
Definition 4. We declare a parking procedure $\mathcal{P}$ to be *local* if it satisfies **Lucky parking, Local move, Shift invariance** and **Local decision**.

All these requirements may not seem completely natural at first sight, but their motivation and use will become clearer in the next sections.

Example 5. The classical parking procedure $\mathcal{P}^{usual}$ is obtained in the standard case $A = \mathbb{Z}$ by always picking the nearest spot to the right.

Example 6. The CS-parking procedure[3] $\mathcal{P}^{CS}$ will be introduced and studied in [5]. We consider the standard case $A = \mathbb{Z}$. To define $\mathcal{P}^{CS}$ inductively, let W, a, I such that I is a maximal interval of $\mathcal{P}^{CS}(W)$ and $a \in I$. Let $j \in \{1, \dots, r\}$ be maximal such that $a_j \in I$ ("last car that parked on the interval"). Then if $a < a_j$ park left of I, and if $a \geq a_j$ park right: this defines $\mathcal{P}^{CS}(Wa)$.

So one needs to remember, for each maximal interval, where the last car that parked there had *wanted* to park. We indicate this with a red cross above certain cars (= blue dots) in the illustration below for the word $W = 5.11.8.3.9.3.2$, which shows $\mathcal{P}^{CS}(W) = \{2, 3, 4, 5, 8, 9, 11\}$.



3 Enumeration

We consider $\mathcal{P}$ any local parking procedure, here and in the rest of this abstract.

Definition 7. A word $a_1 \cdots a_r$ is a *$\mathcal{P}$-parking word* if $\mathcal{P}(a_1 \cdots a_r) = \{1, \dots, r\}$.

Let $Park(\mathcal{P})$ be the set of parking words for $\$\mathcal{P}$. Classical parking functions/words correspond clearly to the case $\mathcal{P}^{usual}$. The case of $\mathcal{P}^{CS}$-parking words is of particular importance in [5], and was one of the motivations for the present abstract.

Here we fix $i \leq r$. Let $A^i_{[r+1]}$ be the set of words of length i and letters with values in $[r+1] = \{1, \dots, r+1\}$.

Cyclic parking We define $\mathcal{P}^{(r)}$ to be the cyclic version of $\mathcal{P}$, as follows: It is defined on $A^i_{[r+1]}$ for any $i \leq r$ and its image is a subset of $\mathbb{Z}/(r+1)\mathbb{Z}$ of size i. Suppose $\mathcal{P}^{(r)}(W)$ is defined for $i < r$, and pick a letter a with $\mathrm{val}(a) \in [r+1]$. If $\mathrm{val}(a) \notin \mathcal{P}^{(r)}(W)$, park it at $\mathrm{val}(a)$. If not, one checks readily that the conditions **Shift invariance** and **Local decision** allow us to determine unambiguously to park left or right, based on any lift of $\mathcal{P}^{(r)}(W)$ to $\mathbb{Z}$, and thus to define $\mathcal{P}^{(r)}(Wa)$.

If $W \in A^i_{[r+1]}$ and $k \in \{1, \dots, r+1\}$, define $W[k] \in A^i_{[r+1]}$ to be the word obtained by replacing each letter (a, s) with $((\tau^k(a) \mod r+1, s)$. We have the following easy lemma:

[3] The full procedure has extra information $S = \mathbb{N}$ and we only allow words on $A = \mathbb{Z} \times \mathbb{N}$ without repeated letters, in the sense of Remark 2. Letters in $A = \mathbb{Z} \times \mathbb{N}$ are then ordered lexicographically for comparison (note that $a = a_j$ does not occur in this setting).

Lemma 8 ("Pollak's argument"). *Let $W \in A^r_{[r+1]}$ and $k \in \{1, \dots, r+1\}$.*
Then $\mathcal{P}^{(r)}(W) = \mathbb{Z}/(r+1)\mathbb{Z} - \{k\}$ if and only if $W[k]$ is in $Park(\mathcal{P})$.

Now for any $\mathcal{G} \subset A^{r+1}_r$, define $\mathcal{G}[k] = \{W[k] \mid W \in \mathcal{G}\}$. From the previous lemma one obtains:

$$\sum_{k=0}^{r} Park(\mathcal{G}[k]) = |\mathcal{G}|. \tag{1}$$

In particular, if $\mathcal{G}$ is stable under cyclic shift, that is $\mathcal{G} = \mathcal{G}[1]$ and thus $\mathcal{G} = \mathcal{G}[k]$ for all k, we have $Park(\mathcal{G}) = \frac{|\mathcal{G}|}{r+1}$. In the standard case $A = \mathbb{Z}$, we can take $\mathcal{G} = \{1, \dots, r+1\}^r$ and get:

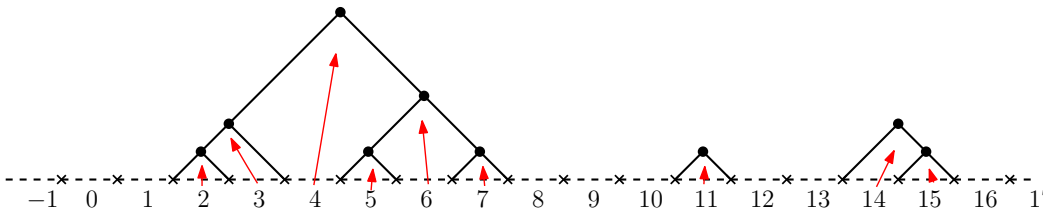
Corollary 9. *For any standard parking procedure $\mathcal{P}$, the number of $\mathcal{P}$-parking words of length r is given by $(r+1)^{r-1}$.*

This shows that the enumeration of parking functions is *universal* for all such standard procedures. For instance, the CS-parking functions are enumerated by $(r+1)^{r-1}$. Note that however there does not seem to be any nice characterization of them, in contrast to the one for usual parking functions recalled in the introduction. More generally, the properties of $\mathcal{P}$-parking functions may differ vastly depending on the procedures, even if their enumeration is the same.

4 Encoding with binary trees

Clearly the parking procedures $\mathcal{P}$ are not injective in general: even if we know π^W, which encodes occupied spots for each prefix of a word W, we cannot reconstruct W. Here we will define a general *lift* of the parking procedure that completely encodes the word W.

Recall that a (finite, plane, binary) *tree* is defined recursively as either empty or consisting of a node, a left subtree and a right subtree. Its size is its number of nodes. These are well-known to be in bijection with *complete* binary trees by attaching extra leaves. A forest is then usually defined as a set of trees. Here we will always mean *indexed* forests, as illustrated below: each binary tree is first completed, and naturally attached to an interval of $\mathbb{Z}$, and these intervals must be the maximal intervals of a finite subset of $\mathbb{Z}$ called the *support* of the forest. The support in the example is $\{2, 3, 4, 5, 6, 7, 11, 14, 15\}$. Elements of the support correspond bijectively to nodes of the forest, as illustrated by the arrows in the figure: this is the *canonical labeling* of the nodes.



Given a procedure $\mathcal{P}$ and a word W, we will recursively attach a pair (P, Q) of labeled forests with the same underlying forest[4] of size r. P will be labeled by the multiset of letters of W, while Q is *decreasing*: it has labels $\{1, \dots, r\}$ and each node has greater label than all its descendants.

Construction Assume $(P', Q') = \widehat{\mathcal{P}}(W)$ where W has size r. By induction we have that the common support of $\widehat{\mathcal{P}}(W)$ is the subset $\mathcal{P}(W)$ of size r. Consider a letter a, and let $i = \mathrm{ls}(Wa)$. We distinguish three cases in order to define P:

1. *$\mathcal{P}(Wa)$ has one more interval than $\mathcal{P}(W)$.* This interval is necessarily the singleton $\{i\}$. In this case add to P' a single node tree canonically labeled by i, and label it by a: this defines P.

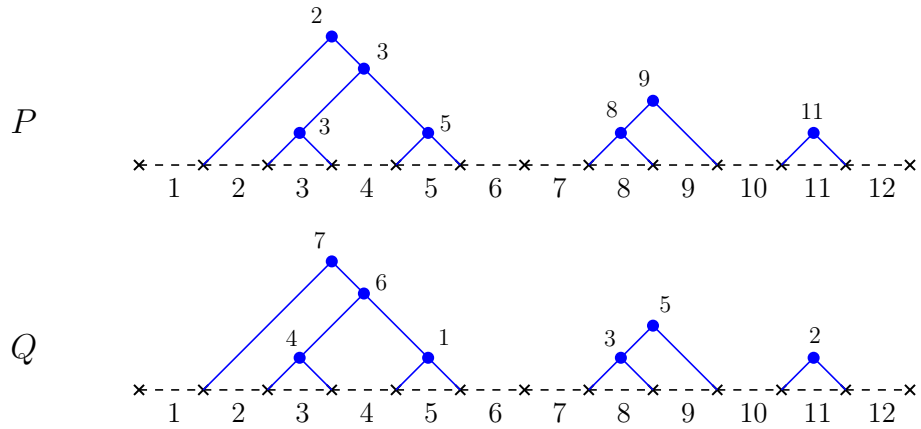
[4] Equivalently, one could of course gather the two labelings on the underlying shape.

2. $\mathcal{P}(Wa)$ *has one fewer interval than* $\mathcal{P}(W)$. So i is adjacent to two intervals of $\mathcal{P}(W)$, corresponding inductively to two trees of $\widehat{\mathcal{P}}(W)$. We add a new root labeled a in P' and attach to it the two subtrees. This gives P, in which the new node has canonical labeling i.
3. $\mathcal{P}(Wa)$ *and* $\mathcal{P}(W)$ *have the same number of intervals.* In this case i is on the left or right of an interval of $\mathcal{P}(W)$, but not adjacent to any other interval. We add a new root labeled a in P' and attach to it the subtree corresponding to the interval: it is attached as a right subtree if i is on the left, and as a left subtree if i is on the right.

In all cases, Q is obtained from Q' by labeling the new node by $r+1$. It is immediate by that Q is decreasing.

Definition 10. The encoding $\widehat{\mathcal{P}} : A^* \mapsto (P, Q)$ is called the $\mathcal{P}$-correspondence.

Here is this correspondence for $\mathcal{P}^{CS}$ with the word $W = 5.11.8.3.9.3.2$ from our previous example:



The encoding $\widehat{\mathcal{P}}$ can be thought of as recording the full history of the parking process. Let us list some immediate properties for any local procedure $\mathcal{P}$:

- A word is $\mathcal{P}$-parking if and only if it results in a single tree with support $\{1, \ldots, r\}$ via the $\mathcal{P}$-correspondence.
- The $\mathcal{P}$-correspondence is injective: the word W can be reconstructed by reading off the labels of P in the increasing order given by the labeling of Q.
- The canonical labeling of a node is the spot where the corresponding car ended up parking.

Note that the conditions **Shift invariance** and **Local decision** are not necessary for this construction and properties. Roughly speaking, they determine what kind of P-labelings may occur in the image.

The correspondence is particularly nice if for any P in the image, all Q with same underlying tree give a pair (P, Q) in the image of $\widehat{\mathcal{P}}$. In that case the image of $\widehat{\mathcal{P}}$ is determined once we know what labeled trees P can occur.

This is the case for our running examples: For $\mathcal{P}^{usual}$, a valid labeling of a node is the canonical label of one of its left descendants (or its own). For $\mathcal{P}^{CS}$, a valid labeling of a node has to be weakly larger than the label of its left child, and strictly smaller than the label of its right child (if such children are not present, use instead the canonical label of the node).

5 Extensions

5.1 Probabilistic parking

To add probabilities to the setting, one can ask how likely it is for a word to be parking. One can also study properties of $\mathcal{P}$-parking words picked uniformly at random for instance. In the classical case this is very natural from the hashing viewpoint, see [2, 3].

It is also possible to probabilize the procedures themselves, by having $\mathcal{P}$ associate to each word W a finite probability measure on $\mathrm{Fin}(\mathbb{Z})$. The local requirements extend naturally to define such procedures, as do most results given in this abstract. One may then ask: what is the probability of a fixed W to be parking ?

Such a procedure is defined at the end of [4]: one flips a (fixed) coin to determine to park left or right. Another possibility is to do a random walk on the interval that is hit, so that parking on the left/right depends on the interval size, and where one desires to park. This leads to the rich combinatorics of *remixed Eulerian numbers*, introduced by the author with Vasu Tewari [6].

5.2 Special subclasses

A natural subclass of parking procedures consists of those where the parking decisions only depend on the set of occupied spots:

Definition 11. A parking procedure $\mathcal{P}$ is called *Markovian* if there exists a function $M : \mathrm{Fin}(\mathbb{Z}) \times A \to \mathrm{Fin}(\mathbb{Z})$ such that $\mathcal{P}(a_1 \cdots a_r a) = M(\mathcal{P}(a_1 \cdots a_r), a)$ for any letters $a_1, \cdots, a_r, a$.

Clearly $\mathcal{P}^{usual}$ is Markovian, while $\mathcal{P}^{CS}$ is not (one needs the red crosses in the figure).

Definition 12. A parking procedure $\mathcal{P}$ is called *abelian* if $\mathcal{P}(a_1 \cdots a_r) = \mathcal{P}(a_{\sigma_1} \cdots a_{\sigma_r})$ for any letters $a_1, \ldots, a_r$ and any permutation σ.

$\mathcal{P}^{usual}$ is abelian, as can be seen immediately from the characterization of parking words in the introduction. $\mathcal{P}^{CS}$ is not, as 112 is parking while 121 is not. It would be interesting to determine other abelian procedures, as these are probably quite rare. One can for instance show that if a standard procedure $\mathcal{P}$ is abelian and Markovian, it is either $\mathcal{P}^{usual}$ or the symmetric "always go left" version.

References

[1] A. Christensen, P. E. Harris, Z. Jones, M. Loving, A. Ramos Rodríguez, J. Rennie, and G. R. Kirby. "A generalization of parking functions allowing backward movement". In: *Electron. J. Combin.* 27.1 (2020), Paper No. 1.33, 18.

[2] P. Diaconis and A. Hicks. "Probabilizing parking functions". In: *Adv. in Appl. Math.* 89 (2017), pp. 125–155. DOI: 10.1016/j.aam.2017.05.004.

[3] P. Flajolet, P. Poblete, and A. Viola. "On the analysis of linear probing hashing". In: *Algorithmica* 22.4 (1998), pp. 490–515. DOI: 10.1007/PL00009236.

[4] A. G. Konheim and B. Weiss. "An Occupancy Discipline and Applications". In: *SIAM J. Appl. Math.* 14.6 (1966), pp. 1266–1274. DOI: 10.1137/0114101.

[5] P. Nadeau and V. Tewari. *Schubert coefficients for the permutahedral variety.* In preparation.

[6] P. Nadeau and V. Tewari. *A q-deformation of an algebra of Klyachko and Macdonald's reduced word formula.* arXiv preprint. 2021. DOI: 10.48550/ARXIV.2106.03828.

[7] R. P. Stanley and J. Pitman. "A polytope related to empirical distributions, plane trees, parking functions, and the associahedron". In: *Discrete Comput. Geom.* 27.4 (2002), pp. 603–634. DOI: 10.1007/s00454-002-2776-6.

[8] C. H. Yan. "Parking functions". In: *Handbook of enumerative combinatorics.* Discrete Math. Appl. (Boca Raton). CRC Press, Boca Raton, FL, 2015, pp. 835–893.

Extremal Sidon sets are Fourier uniform, with applications to partition regularity

Miquel Ortega[1*], Sean Prendiville[2†]

[*]Universitat Politècnica de Catalunya
[†]Dept. of Mathematics and Statistics, Lancaster University

The full version of this work can be found in [16] and will be published elsewhere.

Abstract

Generalising results of Erdős-Freud and Lindström, we prove that the largest Sidon subset of a bounded interval of integers is equidistributed in Bohr neighbourhoods. We establish this by showing that extremal Sidon sets are Fourier-pseudorandom, in that they have no large non-trivial Fourier coefficients. As a further application we deduce that, for any partition regular equation in five or more variables, every finite colouring of an extremal Sidon set has a monochromatic solution.

A subset S of an additively-written abelian group is *Sidon* if every non-zero x has at most one representation as a difference $x = s_1 - s_2$ with $s_1, s_2 \in S$. There have been a number of works investigating the size of Sidon sets $S \subset \mathbb{Z}$. Erdős and Turán [9] established the well-known bound

$$|(n, n+N] \cap S| \leq N^{1/2} + O(N^{1/4}). \tag{1}$$

A corresponding lower bound was found by Singer [18], who constructed a Sidon set $S \subset [N] := \{1, 2, \dots, N\}$ of size

$$|S| \geq N^{1/2} - O(N^{\alpha/2}), \tag{2}$$

where α is a real number for which there is always a prime in $[x - x^{\alpha}, x]$ when x is large (the current record [1] is $\alpha = 0.525$).

Informally, we call a Sidon set $S \subset [N]$ *extremal* if its size is 'close' to $N^{1/2}$ in some sense. There has been speculation on the (im)possibility of characterising such sets [15, 11, 10, 6, 7]. We contribute to this discussion by showing that extremal Sidon sets are Fourier pseudorandom, by which we mean that (after appropriate renormalisation) their Fourier transform behaves essentially like the Fourier transform of the ambient interval.

Definition 1 (Fourier transform)**.** For $f : \mathbb{Z} \to \mathbb{C}$ with finite support define $\hat{f} : \mathbb{T} \to \mathbb{C}$ by

$$\hat{f}(\alpha) := \sum_{n \in \mathbb{Z}} f(n) e(\alpha n).$$

Here $e(\beta)$ stands for $e^{2\pi i \beta}$.

Theorem 2 (Fourier uniformity)**.** *Let $S \subset [N]$ be a Sidon set. Then*

$$\big\| \hat{1}_S - \tfrac{|S|}{N} \hat{1}_{[N]} \big\|_{\infty} \ll N^{1/2} \Big(\Big| \tfrac{|S|}{N^{1/2}} - 1 \Big| + N^{-1/6} \Big)^{1/2}. \tag{3}$$

Remark 3. *The exponent $-1/6$ appearing in* (3) *can be improved to $-1/4$. This is accomplished by replacing a use of* (1) *in our proof with a sharper estimate of Cilleruelo [3].*

[1]Email: miquel.ortega.sanchez-colomer@estudiantat.upc.edu
[2]Email: s.prendiville@lancaster.ac.uk

We note that on combining the Erdős-Turán upper bound (1) with Singer's lower bound (2), the largest Sidon subset S of $[N]$ satisfies $\left||S| - N^{1/2}\right| \ll N^{21/80}$, in which case the $N^{-1/6}$ error term dominates (3).

Corollary 4. *The largest Sidon subset $S \subset [N]$ satisfies*

$$\|\hat{1}_S - \tfrac{|S|}{N}\hat{1}_{[N]}\|_\infty \ll \|\hat{1}_S\|_\infty N^{-\frac{1}{12}}. \tag{4}$$

We are not the first to investigate the uniformity of extremal Sidon sets. Erdős and Freud [8] established that such sets are equidistributed in short intervals[3], whilst Lindström [14] proved equidistribution in arithmetic progressions[4]. We are able to re-prove (quantitatively weaker) versions of these results as a consequence of Theorem 2.

Corollary 5 (Equidistribution in short intervals)**.** *Let $I \subset [0,1]$ be an interval and $S \subset [N]$ a Sidon set of size[5] $|S| \geq \frac{1}{100}N^{1/2}$. Then we have the asymptotic*

$$\mathbb{E}_{x\in S}1_I(x/N) = \mathrm{meas}(I) + O_\varepsilon\Big(N^\varepsilon\Big(\Big|\tfrac{|S|}{N^{1/2}} - 1\Big| + N^{-1/6}\Big)^{1/2}.\Big) \tag{5}$$

Corollary 6 (Equidistribution in residue classes)**.** *For any congruence class $a \pmod q$ and Sidon set $S \subset [N]$ of size $|S| \geq \frac{1}{100}N^{1/2}$ we have the asymptotic*

$$\mathbb{E}_{x\in S}1_{q\cdot\mathbb{Z}+a}(x) = q^{-1} + O_\varepsilon\Big(N^\varepsilon\Big(\Big|\tfrac{|S|}{N^{1/2}} - 1\Big| + N^{-1/6}\Big)^{1/2}.\Big) \tag{6}$$

Theorem 2 is more general than Corollaries 5 and 6, yielding equidistribution in a wider class of sets. In the following, we write $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and $\|\alpha\|_{\mathbb{T}} := \min_{n\in\mathbb{Z}}|\alpha - n|$.

Corollary 7 (Equidistribution in smooth Bohr neighbourhoods)**.** *Let $F : \mathbb{T}^d \to [0,1]$ have Lipschitz constant $K \geq 1$, in that for any $\alpha, \beta \in \mathbb{T}^d$ we have*

$$|F(\alpha) - F(\beta)| \leq K\max_j\|\alpha_j - \beta_j\|_{\mathbb{T}}.$$

Then for any Sidon set $S \subset [N]$ of size $|S| \geq \frac{1}{100}N^{1/2}$ and $\alpha \in \mathbb{T}^d$ we have

$$\mathbb{E}_{x\in S}F(\alpha x) = \mathbb{E}_{x\in[N]}F(\alpha x) + O\Big(K^{\frac{2}{3}}\Big(\Big|\tfrac{|S|}{N^{1/2}} - 1\Big| + N^{-1/6}\Big)^{\frac{1}{8d}}\Big).$$

We can drop the smoothness assumption on the Bohr neighbourhood if we are prepared to assume that it is *regular*.

Definition 8 (Regular Bohr set)**.** Given $\alpha \in \mathbb{T}^d$ and $\rho > 0$, we say that the Bohr set

$$B(\alpha,\rho) := \{x \in [N] : \max_i\|\alpha_i x\| \leq \rho\}$$

is *regular* if for any $|\kappa| \leq \frac{1}{100d}$ we have

$$\Big|\frac{|B(\alpha,(1+\kappa)\rho)|}{|B(\alpha,\rho)|} - 1\Big| \leq 100d|\kappa|.$$

Remark 9. *Bourgain [2] established that regular Bohr sets are ubiquitous; see Tao and Vu [19, Lemma 4.25].*

Corollary 10 (Equidistribution in regular Bohr sets)**.** *Let $B = B(\alpha,\rho)$ be a regular Bohr set with $\alpha \in \mathbb{T}^d$. Then for any Sidon set $S \subset [N]$ of size $|S| \geq \frac{1}{100}N^{1/2}$ we have*

$$\mathbb{E}_{x\in S}1_B(x) = \mathbb{E}_{x\in[N]}1_B(x) + O\Big(d\rho^{-1}\Big(\Big|\tfrac{|S|}{N^{1/2}} - 1\Big| + N^{-1/6}\Big)^{\frac{1}{14d}}\Big).$$

[3]There have since been quantitative improvements in this result, see [3].

[4]For quantitative improvements, see [13].

[5]One could replace the factor 1/100 with any positive absolute constant. This assumption makes our conclusions notationally simpler, and is always satisfied in the range of interest, when $|S| = N^{1/2}(1 + o(1))$ with N large.

Partition regularity over extremal Sidon sets

We offer a further application of Theorem 2 in proving a colouring analogue of a recent result of Conlon, Fox, Sudakov and Zhao [5], this being the original motivation for our paper.

Informally, call a Sidon subset of $[N]$ *dense* if its cardinality is a positive proportion of $N^{1/2}$, say $\frac{1}{100}N^{1/2}$. The authors of [5] show that any dense Sidon subset of $[N]$ contains a non-trivial[6] solution to the equation $x_1 + x_2 + x_3 + x_4 = 4x_5$. The essential features of this equation are that its coefficients sum to zero and that it has at least five variables. The five variable condition cannot be relaxed: every Sidon set lacks non-trivial solutions to the four-variable equation $x_1 - x_2 - x_3 + x_4 = 0$. The assumption that the coefficients sum to zero is also necessary, as we now show.

Proposition 11. *There exists a Sidon subset of $[N]$ with at least $\frac{1}{\sqrt{2}}N^{1/2}(1+o(1))$ elements, and which has no solutions to the equation $x_1 + x_2 + x_3 + x_4 = x_5$.*

Proof. Take an extremal Sidon subset S_0 of $\left[\frac{N-1}{2}\right]$ and set $S := 2 \cdot S_0 + 1$. □

Remark 12. *The above construction can be adapted to show that for any homogeneous linear equation whose coefficients do not sum to zero there exists a dense Sidon set lacking solutions to the equation.*

To be able to accomodate equations whose coefficients do not sum to zero, one may consider a colouring analogue of the problem. In order to introduce such kind of colouring results, let us state Rado's theorem on partition regularity, which solves the problem when the ambient set are the whole integers. For a single equation, it reads as follows.

Theorem 13 (Rado)**.** *Let $c_1, \dots, c_s \in \mathbb{Z} \setminus \{0\}$ such that there exists a non-empty index set $I \subset [s]$ satisfying $\sum_{i\in I} c_i = 0$. There exists $n_0 = n_0(c_1, \dots, c_s)$ such that the following holds. For all $N \geq n_0$ and partition $[N] = C_1 \sqcup \cdots \sqcup C_s$ there exists a solution of*

$$c_1x_1 + \cdots + c_sx_s = 0$$

with $x_i \in C_j$ for a fixed j.

See [12, §3.2] for further details. We usually interpret the C_i as colors of the ambient set and such solutions as *monochromatic* solutions. We call equations satisfying the conclusion of the theorem partition regular.

Compare the previous result to Roth's theorem [17], which is a density type result.

Theorem 14 (Roth)**.** *Let $c_1, \dots, c_s \in \mathbb{Z} \setminus \{0\}$ satisfying $c_1 + \cdots + c_s = 0$ and a fixed δ. There exists $n_0 = n_0(c_1, \dots, c_s, \delta)$ such that the following holds. For all $N \geq n_0$ and $X \subseteq [N]$ with $|X|/N \geq \delta$ there exists a solution of*

$$c_1x_1 + \cdots + c_sx_s = 0$$

with $x_i \in X$.

Therefore, one may speculate on whether a colouring analogue of the results of Conlon, Fox, Sudakov and Zhao [5] should hold. Proposition 11 indicates that one cannot hope to always find monochromatic solutions to $x_1 + x_2 + x_3 + x_4 = x_5$ in colourings of *dense* Sidon sets. In the following theorem we show that such a result does hold for colourings of *extremal* Sidon sets.

Theorem 15 (Partition regularity over extremal Sidon sets)**.** *Let $c_1, \dots, c_s \in \mathbb{Z} \setminus \{0\}$ with $s \geq 5$ and suppose that there exists a non-empty index set $I \subset [s]$ satisfying $\sum_{i\in I} c_i = 0$. Let r be a positive integer and $S \subset [N]$ a Sidon set. Then at least one of the following holds:*

- *N is small, in that $N \ll_{c_1,\dots,c_s,r} 1$.*

[6] For instance, with all variables distinct.

- *S is not extremal, in that* $\left||S| - N^{1/2}\right| \gg_{c_1,\dots,c_s,r} N^{1/2}$.
- *Partition regularity: For any r-colouring $S = C_1 \cup \cdots \cup C_r$, there exists a colour class C_j such that*

$$\sum_{c_1x_1+\cdots+c_sx_s=0} 1_{C_j}(x_1)\cdots 1_{C_j}(x_s) \gg_{c_1,\dots,c_s,r} |S|^s N^{-1}$$

Remark 16. *If $\sum_{i\in I} c_i \neq 0$ for all $\emptyset \neq I \subset [s]$, then there exists a finite colouring of $\mathbb{N}$ with no monochromatic solutions to the equation $c_1x_1 + \cdots + c_sx_s = 0$ (see Rado's criterion for partition regularity [12, §3.2]). Hence the assumption that some subset of coefficients sums to zero is necessary.*

Acknowledgements

We thank David Conlon for an informative talk on [5] in the *Webinar in Additive Combinatorics*[7].

References

[1] R. C. Baker, G. Harman, and J. Pintz. "The difference between consecutive primes. II". In: *Proc. London Math. Soc. (3)* 83.3 (2001), pp. 532–562. DOI: 10.1112/plms/83.3.532.

[2] J. Bourgain. "On triples in arithmetic progression". In: *Geom. Funct. Anal.* 9.5 (1999), pp. 968–984. DOI: 10.1007/s000390050105.

[3] J. Cilleruelo. "Gaps in dense Sidon sets". In: *Integers* 0 (2000), A11, 6pp.

[4] D. Conlon and W. T. Gowers. "Combinatorial theorems in sparse random sets". In: *Ann. of Math. (2)* 184.2 (2016), pp. 367–454. DOI: 10.4007/annals.2016.184.2.2.

[5] D. Conlon, J. Fox, B. Sudakov, and Y. Zhao. *The regularity method for graphs with few 4-cycles*. Preprint. 2020. DOI: 10.48550/ARXIV.2004.10180.

[6] S. Eberhard. *The apparent structure of dense Sidon sets*. Talk given at CANT. https://youtu.be/s4ItIkkUvF4. 2021.

[7] S. Eberhard and F. Manners. *The apparent structure of dense Sidon sets*. Preprint. 2021. DOI: 10.48550/ARXIV.2107.05744.

[8] P. Erdős and R. Freud. "On sums of a Sidon-sequence". In: *J. Number Theory* 38.2 (1991), pp. 196–205. DOI: 10.1016/0022-314X(91)90083-N.

[9] P. Erdös and P. Turán. "On a problem of Sidon in additive number theory, and on some related problems". In: *J. London Math. Soc.* 16 (1941), pp. 212–215. DOI: 10.1112/jlms/s1-16.4.212.

[10] A. Forey and E. Kowalski. *Algebraic curves in their Jacobian are Sidon sets*. Preprint. 2021. DOI: 10.48550/ARXIV.2103.04917.

[11] W. Gowers. *What are dense Sidon subsets of $\{1, 2, \dots, n\}$ like?* Blogspot available at https://bit.ly/3xHq2NM. 2012.

[12] R. L. Graham, B. L. Rothschild, and J. H. Spencer. *Ramsey theory*. Second. Wiley-Interscience Series in Discrete Mathematics and Optimization. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1990, pp. xii+196.

[13] M. N. Kolountzakis. "On the uniform distribution in residue classes of dense sets of integers with distinct sums". In: *J. Number Theory* 76.1 (1999), pp. 147–153. DOI: 10.1006/jnth.1998.2351.

[14] B. Lindström. "Well distribution of Sidon sets in residue classes". In: *J. Number Theory* 69.2 (1998), pp. 197–200. DOI: 10.1006/jnth.1997.2217.

[7] https://sites.google.com/view/web-add-comb/

[15] K. O'Bryant. "A complete annotated bibliography of work related to Sidon sequences". In: *Electron. J. Combin.* DS11.Dynamic Surveys (2004), p. 39.

[16] M. Ortega and S. Prendiville. *Extremal Sidon sets are Fourier uniform, with applications to partition regularity*. Preprint. 2021. DOI: 10.48550/ARXIV.2110.13447.

[17] K. F. Roth. "On certain sets of integers. II". In: *J. London Math. Soc.* 29 (1954), pp. 20–26. DOI: 10.1112/jlms/s1-29.1.20.

[18] J. Singer. "A theorem in finite projective geometry and some applications to number theory". In: *Trans. Amer. Math. Soc.* 43.3 (1938), pp. 377–385. DOI: 10.2307/1990067.

[19] T. Tao and V. Vu. *Additive combinatorics*. Vol. 105. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2006, pp. xviii+512. DOI: 10.1017/CBO9780511755149.

Many regular triangulations and many polytopes

Arnau Padrol[1*], Eva Philippe[2*], Francisco Santos[3†]

[*]Sorbonne Université, CNRS, IMJ-PRG, F-75005 Paris, France
[†]Dept. de Matemáticas, Estadística y Computación, Universidad de Cantabria, Santander, Spain

Work of Padrol and Philippe is supported by grants ANR-17-CE40-0018 and ANR-21-CE48-0020 of the French National Research Agency ANR (projects CAPPS and PAGCAP). Work of Santos is supported by grant PID2019-106188GB-I00 of MCIN/AEI/10.13039/501100011033 and project CLaPPo (21.SI03.64658) of Universidad de Cantabria and Banco Santander.

Abstract

We show that for fixed $d > 3$ and n growing to infinity there are at least $(n!)^{d-2\pm o(1)}$ different labeled combinatorial types of d-polytopes with n vertices. This is about the square of the previous best lower bounds. As an intermediate step, we show that certain neighborly polytopes (such as particular realizations of cyclic polytopes) have at least $(n!)^{\lfloor \frac{d-1}{2} \rfloor \pm o(1)}$ regular triangulations.

1 Introduction

A *polytope* is a geometric object defined as the convex hull of a finite number of points in a real Euclidean space. Its *combinatorial type* is given by its poset of *faces* (subsets of the polytope maximized by linear functionals, ordered by inclusion). Enumerating or counting the different combinatorial types of polytopes is a hard problem that goes back to Cayley and Kirkman in the nineteenth century. The problem remains hard even when restricting to the "generic" case of *simplicial* polytopes, where all faces except the whole polytope are simplices (equivalently, polytopes whose combinatorial type does not change when the vertices are perturbed). See the historical remarks by Grünbaum in [6, Preface & Chapter 13.6], where the difficulty of this question is taken as one of the main reasons for a "*decline in the interest in convex polytopes*" around the turn of the century.

In 1986 Goodman and Pollack [5] showed that the number of (labeled) combinatorially different simplicial d-polytopes with n vertices is relatively small, bounded by $(n!)^{c_d}$ for some constant c_d depending solely on d, and Alon [1] proved that this upper bound is valid for non-necessarily simplicial polytopes too. This contrasts with the number of combinatorially different simplicial $(d-1)$-spheres with n vertices, which grows at least as $e^{\Omega(n^{\lfloor d/2 \rfloor})}$ [7, 9].

In 1982 Shemer [11] had devised constructions producing $(n!)^{\frac{1}{2}\pm o(1)}$ different simplicial polytopes. This matches the upper bounds, except for the fact that the constant c_d in the upper bound of Goodman and Pollack and Alon is $d^2 \pm o(1)$, much bigger than the $1/2$ obtained by Shemer. The lower bound was greatly improved by Padrol [10] (see also [4]) who showed that there are at least $(n!)^{\lfloor d/2 \rfloor \pm o(1)}$ (labeled) neighborly polytopes. The main result in this paper is that we essentially double the exponent of $n!$ in this lower bound:

Theorem 1. *The number of different labeled combinatorial types of d-polytopes with n vertices for fixed $d > 3$ and n growing to infinity is at least $(n!)^{d-2\pm o(1)}$.*

[1]Email: arnau.padrol@imj-prg.fr
[2]Email: eva.philippe@imj-prg.fr
[3]Email: francisco.santos@unican.es

Enumerating polytopes is intimately tied to enumerating *regular* triangulations of point configurations, which are those arising as lower envelopes of polytopes of one dimension more; in fact, the number of (combinatorial types of) simplicial d-polytopes with n vertices coincides with that of $(d-1)$-dimensional regular triangulations with $n-1$ vertices. See the beginning of Section 4 for details on this relation. In the same vein, counting all triangulations, regular or not, is related to counting simplicial spheres.

In particular, the Goodman-Pollack bound implies the same upper bound of $(n!)^{d^2 \pm o(1)}$ for the number of regular triangulations, while the construction of Kalai [7] can be adapted to derive that the cyclic d-polytope with n vertices has at least $e^{\Omega(n^{\lfloor d/2 \rfloor})}$ triangulations in total [3, Theorem 6.1.2].

Observe that the upper bound is for the total number of (combinatorially different) regular triangulations of *all* polytopes (for fixed parameters n and d), while the construction of Kalai counts triangulations of *a single polytope*. For regular triangulations of a single polytope, it is shown in [3, Theorem 7.2.10] that the Cartesian product of a cyclic 3-polytope with n vertices and a segment has at least $(n/2)! = (n!)^{1/2 \pm o(1)}$ regular triangulations. The second result in this paper is a significant improvement of this lower bound, showing for example that:

Theorem 2. *For fixed $d \geq 3$ and n going to infinity, there are realizations of the cyclic d-polytope with n vertices having at least*

$$(n!)^{\lfloor \frac{d-1}{2} \rfloor \pm o(1)}$$

regular triangulations.

It has to be noted that the total number of triangulations of a polytope (or point configuration) depends only on its *oriented matroid* (another combinatorial invariant that is finer than the combinatorial type), while the number of regular triangulations varies for different realizations of the same oriented matroid.

Apart of its intrinsic interest, Theorem 2 is an intermediate step for Theorem 1; the proof of Theorem 1 consists in showing that *all* of the many polytopes constructed by Padrol [10] admit realizations with the many regular triangulations stated in Theorem 2.

2 Definitions and notation

We will follow [12] and [3] for all the terminology concerning convex polytopes, point configurations, and triangulations, and we refer the reader to these references for thorough introduction to the topic.

A *point configuration* is an ordered sequence $P = (p_1, \dots, p_n) \in \mathbb{R}^{d \times n}$. In this paper we will usually assume that the points are distinct and *in convex position*, where the latter means that all of them are vertices of the polytope $\operatorname{conv}(P)$. We say that P is *k-neighborly* if any subset of k points is the vertex set of a face of $\operatorname{conv}(P)$, and just *neighborly* if it is $\left\lfloor \frac{d}{2} \right\rfloor$-neighborly; the latter makes sense since the simplex is the only d-polytope that is more than $\left\lfloor \frac{d}{2} \right\rfloor$-neighborly.

A *triangulation* T of P is a simplicial complex on the set $[n]$ such that

(i) $\bigcup_{F \in T} \operatorname{conv}(\{p_i \,|\, i \in F\}) = \operatorname{conv}(P)$,

(ii) for all $F, F' \in T$, $\operatorname{conv}(\{p_i \,|\, i \in F\}) \cap \operatorname{conv}(\{p_i \,|\, i \in F'\})$ is a common face of $\operatorname{conv}(\{p_i \,|\, i \in F\})$ and $\operatorname{conv}(\{p_i \,|\, i \in F'\})$.

A triangulation T of P is *regular* if there is a lifting vector $w \in \mathbb{R}^{[n]}$ such that for any $F \in T$, $\operatorname{conv}(\{(p_i, w_i) \,|\, i \in F\})$ is a lower face of $\operatorname{conv}(\{(p_i, w_i) \,|\, i \in [n]\})$. Here, we call a face F of a polytope P in $\mathbb{R}^{d+1}$ *lower* if its outer normal cone contains a vector with last coordinate negative. That is, if there is a functional $c \in \mathbb{R}^{d+1}$ with $c_{d+1} < 0$ that is maximized on F.

We denote $\mathcal{T}(P)$ the set of regular triangulations of P. For a triangulation T, we call *cells* its maximal faces. (These are sometimes called facets, but we reserve the word *facet* for facets of a polytope).

A point $q \notin P$ is said to be in *general position* with respect to P if no hyperplane spanned by points of P contains q, and in *very general position* with respect to P if moreover any small perturbation of q does not change $\mathcal{T}(P \cup \{q\})$. An argument similar to that in [2, Part 2] shows that configurations in very general position form a dense open subset of the space of all point configurations.

Two points $p_i, p_j \in P$ are said to be *triangulation-inseparable* in P if we have that

(i) $\mathcal{T}(P \setminus \{p_i\}) = \mathcal{T}(P \setminus \{p_j\})$ up to relabeling j to i, and

(ii) for any $T \in \mathcal{T}(P \setminus \{p_i\})$ there is a lifting vector $w \in \mathbb{R}^{[n]}$ which restricted to both $P \setminus \{p_i\}$ and $P \setminus \{p_j\}$ produces T as a regular triangulation.

Let p be a vertex of $\mathrm{conv}(P)$. We define P/p to be any point configuration obtained as the intersection of the half-lines positively spanned by $\{p' - p \,|\, p' \in P \setminus \{p\}\}$ with an affine hyperplane that does not contain p and intersects all these half-lines. Following [3, Definition 4.2.9] we call P/p the *contraction* of P at the point p. All the configurations that can be obtained as P/p have the same triangulations and the same regular triangulations. In fact, regular triangulations of P/p are exactly the links at p of regular triangulations of P [3, Lemmas 4.2.20 and 4.2.22]. Here, the *link* of a triangulation T at a point p_i, which we denote T/p_i, is defined as

$$T/p_i := \{F \subset [n] \setminus \{i\} \,|\, F \cup \{i\} \in T\}\,.$$

3 Many regular triangulations

The main idea of our construction of configurations with a large number of regular triangulations is to split a point into two triangulation-inseparable points and to estimate the number of regular triangulations generated after this operation.

Lemma 3. *Let P be a point configuration in convex position in $\mathbb{R}^d$ and $p \in P$ in very general position with respect to $P \setminus \{p\}$. We denote C the minimum number of cells in a regular triangulation of P/p.*

Then for every p' such that p and p' are triangulation-inseparable in $P \cup \{p'\}$ we have

$$|\mathcal{T}(P \cup \{p'\})| \geq |\mathcal{T}(P)| \times (C+1).$$

Without any further constraint this is not very useful, as $\mathrm{conv}(P/p)$ could be a simplex and $C = 1$. However, a lower bound on C can be proved if we have knowledge on the neighborliness of P/p, thanks to the following lemma whose proof follows from a result of McMullen and Walkup [8, Thm. 2].

Recall that for a d-dimensional simplicial complex $\mathcal{C}$ and $0 \leq j \leq d+1$ we denote

$$h_j(\mathcal{C}) = \sum_{k=0}^{j} (-1)^{j-k} \binom{d+1-k}{d+1-j} f_{k-1}(\mathcal{C}),$$

where $f_k(\mathcal{C})$ is the number of faces of $\mathcal{C}$ of dimension k.

Lemma 4. *Let $d > 2$ and $1 \leq k \leq d+1$. Let Q be a d-dimensional simplicial polytope on n vertices. Then the number of cells in any triangulation of Q is bounded below by $h_k(\partial Q)$.*

In particular, if Q is k-neighborly for $1 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor$, then this number is bounded by:

$$h_k(\partial Q) = \binom{n-d-1+k}{k}.$$

As a consequence of the previous two lemmas we have:

Theorem 5. *Let $P = (p_1, \ldots, p_{n-1}, q)$ be a configuration of n points in convex position in $\mathbb{R}^d$ such that:*

(i) for every $d+1 \le i \le n-1$, p_i and q are triangulation-inseparable in $P_i = (p_1, \ldots, p_i, q)$, and

(ii) the point configuration P/q is k-neighborly.

Then

$$|\mathcal{T}(P)| \geq \Pi_{m=d+1}^{n} \binom{m-d-1+k}{k},$$

which is of order $(n!)^{k \pm o(1)}$.

The following lemma shows that we can easily construct triangulation-inseparable pairs.

Lemma 6. *Let P be a point configuration in $\mathbb{R}^d$ and $p \in P$ in very general position with respect to $P \setminus \{p\}$. Then there is an $\varepsilon > 0$ such that p and p' are triangulation-inseparable in $P \cup \{p'\}$ for any $p' \in B(p, \varepsilon)$ in very general position with respect to P. Here $B(p, \varepsilon)$ denotes the ball of radius ε centered at p.*

The combination of these results provides our bound for the number of regular triangulations of cyclic polytopes. Indeed, by sliding vertices along the moment curve, we can put the last vertices of the cyclic polytope arbitrarily close which, by Lemma 6, implies them to be triangulation-inseparable after a perturbation into very general position if needed. Moreover, the vertex figure of the last vertex in a cyclic polytope is a $(d-1)$-dimensional cyclic polytope, and therefore $\left\lfloor \frac{d-1}{2} \right\rfloor$-neighborly. Hence Theorem 5 gives:

Corollary 7. *There are realizations of the cyclic polytope of dimension d with n vertices that admit at least $(n!)^{\lfloor \frac{d-1}{2} \rfloor \pm o(1)}$ regular triangulations.*

4 Many polytopes

Let us call *polytopal d-ball* any (labeled) simplicial complex that can be realized as a regular triangulation of a configuration of points in dimension d. By adding a point "at infinity" to a polytopal d-ball one obtains a polytopal d-sphere with one more vertex, and viceversa. Thus, the number of combinatorially different labeled polytopal d-balls with n vertices coincides with the number of combinatorially different labeled simplicial $(d+1)$-polytopes with $n+1$ vertices.

On the other hand, if two simplicial polytopes are combinatorially different then no triangulation of the first can be combinatorially equal to one of the second, because we can recover the boundary complex of a simplicial polytope from any of its triangulations. Hence:

Lemma 8. *If $P_1, \ldots, P_N$ are configurations of dimension d and size n in convex and general position and with combinatorially different convex hulls, then there are at least*

$$\sum_{i=1}^{N} |\mathcal{T}(P_i)|$$

combinatorially different labeled simplicial $(d+1)$-polytopes with $n+1$ vertices.

In this section we show that not only cyclic polytopes but all the *Gale sewn* polytopes introduced in [10] fulfill (in certain realizations) the conditions of Theorem 5. This provides us with a large family of polytopes with many regular triangulations, to which we can apply Lemma 8 and obtain even more polytopes.

A central tool for the construction are lexicographic liftings, which are a way to derive $(d+1)$-dimensional point configurations from d-dimensional point configurations.

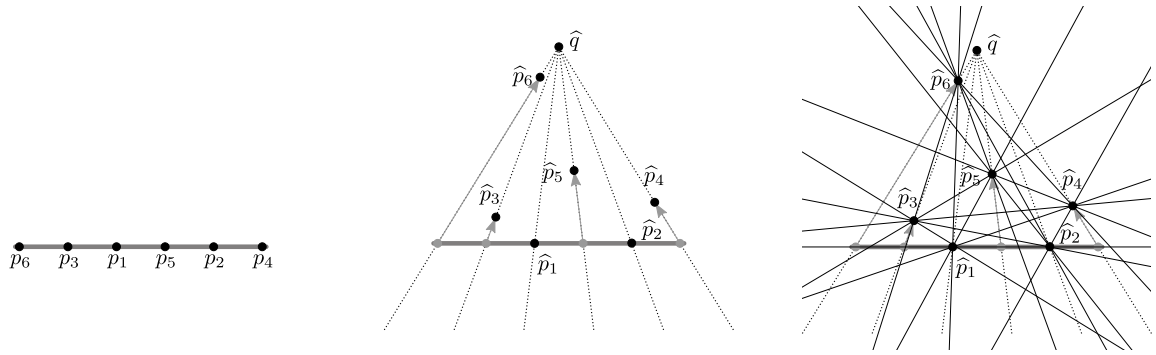
Definition 9. A *positive lexicographic lifting* of a point configuration $P = (p_1, \dots, p_n) \subset \mathbb{R}^d$ (with respect to the order induced by the labels) is any configuration $\widehat{P} = (\widehat{p}_1, \dots, \widehat{p}_n, \widehat{q})$ of $n+1$ labeled points in $\mathbb{R}^{d+1}$ such that:

(i) $\widehat{q}$ is a point outside the hyperplane $x_{d+1} = 0$,

(ii) for $1 \le i \le n$, the point $\widehat{p}_i$ lies in the half-line from $\widehat{q}$ through $(p_i, 0)$,

(iii) for $d+2 \le i \le n$, and for every hyperplane H spanned by $d+1$ points taken among $\{\widehat{p}_1, \dots, \widehat{p}_{i-1}\}$, the points $\widehat{q}$ and $\widehat{p}_i$ lie at the same side of H.

Remark 10. *Positive lexicographic liftings exist for every point configuration, and are a special case of the lexicographic liftings defined with a sign vector in $\{+,-\}^n$, defined e.g. in [4, Def. 4.1]. One way to construct a positive lexicographic lifting is to choose $\widehat{q}$ arbitrarily outside the hyperplane $x_{d+1} = 0$ and then take $\widehat{p}_i := (1-\varepsilon_i)\widehat{q} + \varepsilon_i(p_i, 0)$ for constants $0 < \varepsilon_n \ll \varepsilon_{n-1} \ll \cdots \ll \varepsilon_1$. See Figure 1.*



(a) P (b) $\widehat{P}$ (c) Check ((iii))

Figure 1: A positive lexicographic lifting $\widehat{P} \subset \mathbb{R}^2$ of a configuration $P \subset \mathbb{R}^1$.

If one starts with a 0-dimensional point configuration (that is a point repeated multiple times), and then perfoms a sequence of positive lexicographic liftings always with respect to the same order, then one obtains a cyclic polytope. If the order is altered at each step, then many combinatorial types of polytopes are obtained, but not necessarily neighborly. Moreover, different lifting orders might give rise to equivalent polytopes. However, if one restricts to changing the order of the lifting only every two dimensions, then neighborliness is preserved and the combinatorial type can be controlled. This is used in [10] to construct many neighborly polytopes. The original presentation in [10] is in terms of lexicographic extensions of the Gale dual. A primal presentation in terms of lexicographic liftings is given in [4].

Theorem 11 ([10, Theorem 4.2])**.** *Let P be a k-neighborly point configuration in general position. Let $\widehat{P}$ be a lexicographic lifting of P and let $\widehat{\widehat{P}}$ be a positive lexicographic lifting of $\widehat{P}$, with respect to the same order. Then $\widehat{P}$ is k-neighborly and $\widehat{\widehat{P}}$ is $(k+1)$-neighborly.*

Theorem 12 ([10, Theorem 6.8])**.** *The number of labeled combinatorial types of neighborly d-polytopes with n vertices obtained from a 0-dimensional point configuration by a sequence of positive lexicographic liftings (with orders that might change along each step of the sequence) is at least*

$$(n!)^{\lfloor \frac{d}{2} \rfloor \pm o(1)}.$$

The combination of these constructions allows us to prove Theorem 1:

Sketch of proof of Theorem 1. The last step of the construction of the many polytopes of Theorem 12 in even dimension is a positive lexicographic lifting $\widehat{P} = (\widehat{p}_1, \dots, \widehat{p}_n, \widehat{q})$ from a $\left\lfloor \frac{d-1}{2} \right\rfloor$-neighborly polytope P of dimension $d-1$. For odd dimension we can do an extra lexicographic lifting (instead of taking a pyramid as in [10, Corollary 6.10]). Hence, regardless of the parity of the dimension, we have at least $(n!)^{\lfloor d/2 \rfloor \pm o(1)}$ distinct polytopes obtained as positive lexicographic liftings of $\left\lfloor \frac{d-1}{2} \right\rfloor$-neighborly $(d-1)$-polytopes.

Lemma 6 ensures that we can do this lifting step by step so that for every i from $d+1$ to n, $\widehat{p}_i$ and $\widehat{q}$ are triangulation-inseparable in $(\widehat{p}_1, \dots, \widehat{p}_i, \widehat{q})$. Moreover, note that by construction P is the contraction $\widehat{P}/\widehat{q}$, and that similarly $(p_1, \dots, p_i) = (\widehat{p}_1, \dots, \widehat{p}_i, \widehat{q})/\widehat{q}$. These contractions are thus $\left\lfloor \frac{d-1}{2} \right\rfloor$-neighborly.

Hence Theorem 5 applies: each of these polytopes has at least $n^{\left\lfloor \frac{d-1}{2} \right\rfloor n \pm o(n)}$ regular triangulations. Then Lemma 8 gives us a lower bound of $n^{(d-1)n \pm o(n)}$ labeled simplicial types of $(d+1)$-polytopes with $n+1$ vertices. □

References

[1] N. Alon. "The number of polytopes, configurations and real matroids". In: *Mathematika* 33.1 (1986), pp. 62–71.

[2] C. A. Athanasiadis. "The largest intersection lattice of a discriminantal arrangement". In: *Beiträge Algebra Geom.* 40.2 (1999), pp. 283–289.

[3] J. A. De Loera, J. Rambau, and F. Santos. *Triangulations: Structures for Algorithms and Applications*. Vol. 25. Algorithms and Computation in Mathematics. Springer Verlag, 2010.

[4] B. Gonska and A. Padrol. "Neighborly inscribed polytopes and Delaunay triangulations". In: *Adv. Geom.* 16.3 (2016), pp. 349–360. DOI: 10.1515/advgeom-2015-0045.

[5] J. E. Goodman and R. Pollack. "Upper bounds for configurations and polytopes in $\mathbf{R}^d$". In: *Discrete Comput. Geom.* 1.3 (1986), pp. 219–227. DOI: 10.1007/BF02187696.

[6] B. Grünbaum. *Convex polytopes*. Second. Vol. 221. Graduate Texts in Mathematics. Springer-Verlag, New York, 2003, pp. xvi+468. DOI: 10.1007/978-1-4613-0019-9.

[7] G. Kalai. "Many triangulated spheres". In: *Discrete Comput. Geom.* 3.1 (1988), pp. 1–14. DOI: 10.1007/BF02187893.

[8] P. McMullen and D. W. Walkup. "A generalized lower-bound conjecture for simplicial polytopes". In: *Mathematika* 18.2 (1971), pp. 264–273.

[9] E. Nevo, F. Santos, and S. Wilson. "Many triangulated odd-dimensional spheres". In: *Math. Ann.* 364.3-4 (2016), pp. 737–762. DOI: 10.1007/s00208-015-1232-x.

[10] A. Padrol. "Many neighborly polytopes and oriented matroids". In: *Discrete Comput. Geom.* 50.4 (2013), pp. 865–902. DOI: 10.1007/s00454-013-9544-7.

[11] I. Shemer. "Neighborly polytopes". In: *Israel J. Math.* 43.4 (1982), pp. 291–314. DOI: 10.1007/BF02761235.

[12] G. M. Ziegler. *Lectures on Polytopes*. Vol. 152. Graduate texts in Mathematics. New York: Springer-Verlag, 1995, pp. x+370. DOI: 10.1007/978-1-4613-8431-1.

Deformation cones of graphical zonotopes

Arnau Padrol[1*], Vincent Pilaud[2†], Germain Poullot[3*]

[*]Institut de Mathématiques de Jussieu - Paris Rive Gauche, Sorbonne Université, Paris, France
[†]CNRS & LIX, École Polytechnique, Palaiseau, France

The full version of this work can be found in the preprint [16]

Abstract

We study deformations of graphical zonotopes. Deformations of the classical permutahedron (which is the graphical zonotope of the complete graph) have been intensively studied in recent years under the name of generalized permutahedra. We provide an irredundant description of the deformation cone of the graphical zonotope associated to a graph G, consisting of independent equations defining its linear span (in terms of non-cliques of G) and of the inequalities defining its facets (in terms of common neighbors of neighbors in G). In particular, we deduce that the faces of the standard simplex corresponding to induced cliques in G form a linear basis of the deformation cone, and that the deformation cone is simplicial if and only if G is triangle-free.

1 Introduction

The *graphical zonotope* of a graph G is a convex polytope Z_G whose geometry encodes several combinatorial properties of G. For example, its vertices are in bijection with the acyclic orientations of G [24, Prop. 2.5] and its volume is the number of spanning trees of G [25, Ex. 4.64]. When G is the complete graph K_n, the graphical zonotope is a translation of the classical n-dimensional *permutahedron*. This polytope, obtained as the convex hull of the $n!$ permutations of the vector $(1, 2, \dots, n) \in \mathbb{R}^n$, was first introduced by Schoute in 1911 [22], and has become one of the most studied polytopes in geometric and algebraic combinatorics.

A *deformed permutahedron* (*a.k.a.* *generalized permutahedron*) is a polytope obtained from the permutahedron by translating its facets-defining hyperplanes without passing through any vertices. In particular, graphical zonotopes are deformed permutahedra. These polytopes were originally introduced by Edmonds in 1970 under the name of *polymatroids* as a polyhedral generalization of matroids in the context of linear optimization [7]. They were rediscovered by Postnikov in 2009 [19], who initiated the investigation of their rich combinatorial structure. They have since become a widely studied family of polytopes that appears naturally in several areas of mathematics, such as algebraic combinatorics [1, 2, 20], optimization [8], game theory [5], statistics [14, 15], and economic theory [9]. The set of deformed permutahedra can be parametrized by the cone of *submodular functions* [7, 19].

In general, a *deformation* of a polytope P can be equivalently described as (i) a polytope obtained from P by parallel translation of its facets [19, 20], (ii) a polytope obtained from P by moving the vertices so that the directions of all edges are preserved [19, 20], (iii) a polytope whose normal fan coarsens the normal fan of P [11], (iv) a polytope whose support function is a convex piecewise linear continuous function supported on the normal fan of P [4, Sec. 6.1][6, Sec. 9.5], or (v) a Minkowski summand of a dilate of P [13, 23]. The deformations of P form a polyhedral cone under dilation and Minkowski addition, called the *deformation cone* of P [19]. Its interior is the *type cone* of the normal

[1]Email: arnau.padrol@imj-prg.fr. Supp. by ANR CAPPS 17 CE40 0018.
[2]Email: vincent.pilaud@lix.polytechnique.fr. Supp. by ANR CAPPS 17 CE40 0018 and CHARMS 19 CE40 0017.
[3]Email: germain.poullot@imj-prg.fr. Supp. by ANR CAPPS 17 CE40 0018.

fan of P [11], and contains those polytopes with the same normal fan as P. When P has rational vertex coordinates, then the type cone is known as the *numerically effective cone* and encodes the embeddings of the associated toric variety into projective space [4].

There exist several methods to parametrize and describe the deformation cone of a given polytope (see *e.g.* [20, App. 15]), for example via the *height deformation space* and the *wall-crossing inequalities* or via the *edge deformation space* and the *polygonal face equations*. However, these methods only provide redundant inequality descriptions of the deformation cone. Not even the dimension of the deformation cone is easily deduced from these descriptions, as illustrated by the difficulty of describing which fans have a nonempty type cone (*i.e.* describing *realizable fans* [6, Chap. 9.5.3]), or a one dimensional type cone (*i.e.* describing *Minkowski indecomposable* polytopes [12, 13, 21, 23]).

In this extended abstract, we study the deformation cones of arbitrary graphical zonotopes. It is worth noting that most of the existing approaches to compute deformation cones only focus on simple polytopes with simplicial normal fans [3, 20]. Nevertheless, most graphical zonotopes are not simple. They are simple only for chordful graphs (those where every cycle induces a clique), see [20, Prop. 5.2] and [10, Rmk. 6.2]. We thus use an alternative approach to describe the deformation cone of a non-simple polytope based on a simplicial refinement of its normal cone.

Since graphical zonotopes are deformed permutahedra, their deformation cones appear as faces of the submodular cone. However, faces of the submodular cone are far from being well understood: determining its rays for instance remains an open problem since the 1970s [7]. We provide complete irredundant descriptions of deformation cones of graphical zonotopes, derive their dimensions and characterize those which are simplicial. We also obtain that the faces of the standard simplex contained in these deformation cones provide a linear basis of their vector span, generalizing [2].

2 Graphical zonotopes

Let $G \coloneqq (V, E)$ be a graph with vertex set V and edge set E. Let $(\boldsymbol{e}_v)_{v\in V}$ denote the canonical basis of $\mathbb{R}^V$. The *graphical arrangement* $\mathcal{A}_G$ is the arrangement of the hyperplanes $\left\{\boldsymbol{x} \in \mathbb{R}^V \;\middle|\; \boldsymbol{x}_u = \boldsymbol{x}_v\right\}$ for all edges $\{u, v\} \in E$. The *graphical fan* $\mathcal{G}_G$ is the fan whose cones are all the possible intersections of one of the sets $\left\{\boldsymbol{x} \in \mathbb{R}^V \;\middle|\; \boldsymbol{x}_u = \boldsymbol{x}_v\right\}$, $\left\{\boldsymbol{x} \in \mathbb{R}^V \;\middle|\; \boldsymbol{x}_u \geq \boldsymbol{x}_v\right\}$, or $\left\{\boldsymbol{x} \in \mathbb{R}^V \;\middle|\; \boldsymbol{x}_u \leq \boldsymbol{x}_v\right\}$ for each edge $\{u, v\} \in E$. The *graphical zonotope* Z_G is the Minkowski sum $\mathsf{Z}_G \coloneqq \sum_{(u,v)\in E}[\boldsymbol{e}_u, \boldsymbol{e}_v]$ of the line segments $[\boldsymbol{e}_u, \boldsymbol{e}_v] \subseteq \mathbb{R}^V$ for all edges $\{u, v\} \in E$. The normal fan of Z_G is $\mathcal{G}_G$. Note that the lineality space of $\mathcal{G}_G$ is the subspace $\mathbb{K}_G$ of $\mathbb{R}^V$ spanned by the characteristic vectors of the connected components of G, and that Z_G lies in a subspace orthogonal to $\mathbb{K}_G$.

An *ordered partition* (μ, ω) of G consists of a partition μ of V where each part induces a connected subgraph of G, together with an acyclic orientation ω of the quotient graph G/μ. It corresponds to the cone $\mathsf{C}_{\mu,\omega}$ of $\mathcal{G}_G$ defined by the inequalities $\boldsymbol{x}_u \leq \boldsymbol{x}_v$ for all $u, v \in V$ such that there is a directed path in ω from the part containing u to the part containing v (hence, $\boldsymbol{x}_u = \boldsymbol{x}_v$ if u, v are in the same part of μ). In particular:

- The maximal cones of $\mathcal{G}_G$ are in bijection with the acyclic orientations of G.
- The minimal cones of $\mathcal{G}_G$ (*i.e.* the rays of $\mathcal{G}_G/\mathbb{K}_G$) are in bijection with the *biconnected subsets* of G, *i.e.* non-empty connected subsets of V whose complements in their connected components are also non-empty and connected.
- The rays of $\mathcal{G}_G/\mathbb{K}_G$ that belong to the cone $\mathsf{C}_{\mu,\omega}$ of an ordered partition (μ, ω) of G are the biconnected sets of G that contracted by μ give rise to an upper set of ω.

When G is the complete graph K_n, the graphical fan is the *braid fan* $\mathcal{B}_n$ and the graphical zonotope is the *permutahedron*. In $\mathcal{B}_n$, the faces correspond to ordered partitions of $[n]$, the rays to all proper subsets of $[n]$, and the maximal cones to all permutations of $[n]$.

3 Deformation cones of polytopes

Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a polytope with normal fan $\mathcal{F}$. We consider the *deformation cone* $\mathbb{DC}(\mathsf{P})$ formed by all polytopes whose normal fans coarsen $\mathcal{F}$ (alternative definitions were recalled in the introduction). Note that $\mathbb{DC}(\mathsf{P})$ is a closed convex cone (dilations and Minkowski sums preserve deformations) and contains a lineality subspace of dimension d (translations preserve deformations). Its interior, called the *type cone* of $\mathcal{F}$ by P. McMullen [11], consists of all polytopes whose normal fan is $\mathcal{F}$. Taking into account the lineality, we say that the deformation cone is *simplicial* when its quotient modulo translations is simplicial, and we call *rays* of $\mathbb{DC}(\mathsf{P})$ the rays of its quotient modulo translations. They are spanned by the Minkowski indecomposable deformations of P of dimension at least 1 (note that 0-dimensional deformations account for the space of translations).

There are several linearly isomorphic presentations of the deformation cone [11, 13, 20]. We will use the following convenient formulation [18, Prop. 3] adapted from the classical *wall-crossing inequalities* [3, Lem. 2.1]. To deal with non-simple polytopes as well, it uses a simplicial refinement of the normal fan. If the refinement contains additional rays, then the type cone is embedded in a higher dimensional space, but these additional coordinates can just be projected out. We say that a fan $\mathcal{F}$ is *supported* on the set of vectors $\boldsymbol{S}$ if every cone of $\mathcal{F}$ is spanned by a subset of $\boldsymbol{S}$.

Proposition 1. *Let $\mathsf{P} \subseteq \mathbb{R}^d$ be a polytope whose normal fan $\mathcal{F}$ is refined by the simplicial fan $\mathcal{G}$ supported on $\boldsymbol{S}$. Then the deformation cone $\mathbb{DC}(\mathsf{P})$ of P is the set of all polytopes constructed as $\left\{\boldsymbol{x} \in \mathbb{R}^d \;\middle|\; \langle\, \boldsymbol{s} \mid \boldsymbol{x} \,\rangle \le \boldsymbol{h}_{\boldsymbol{s}} \text{ for all } \boldsymbol{s} \in \boldsymbol{S}\right\}$ for all $\boldsymbol{h}$ in the cone of $\mathbb{R}^{\boldsymbol{S}}$ defined by*

*(i) the equalities $\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, \boldsymbol{h}_{\boldsymbol{s}} = 0$ for any adjacent maximal cones $\mathbb{R}_{\ge 0}\boldsymbol{R}$ and $\mathbb{R}_{\ge 0}\boldsymbol{R}'$ of $\mathcal{G}$ belonging to **the same** maximal cone of $\mathcal{F}$,*

*(ii) the inequalities $\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, \boldsymbol{h}_{\boldsymbol{s}} \ge 0$ for any adjacent maximal cones $\mathbb{R}_{\ge 0}\boldsymbol{R}$ and $\mathbb{R}_{\ge 0}\boldsymbol{R}'$ of $\mathcal{G}$ belonging to **distinct** maximal cones of $\mathcal{F}$,*

where $\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, \boldsymbol{s} = \boldsymbol{0}$ is the unique linear dependence with $\alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{r}) + \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{r}') = 2$ among the rays of two adjacent maximal cones $\mathbb{R}_{\ge 0}\boldsymbol{R}$ and $\mathbb{R}_{\ge 0}\boldsymbol{R}'$ of $\mathcal{F}$ with $\boldsymbol{R} \smallsetminus \{\boldsymbol{r}\} = \boldsymbol{R}' \smallsetminus \{\boldsymbol{r}'\}$.

4 Deformation cones of graphical zonotopes

Common simplicial refinement It is worth noting that most graphical zonotopes are not simple (they are simple only for chordful graphs, where every cycle induces a clique [20, Prop. 5.2]). To describe the deformation cones of non-simple graphical zonotopes, we thus use a simplicial refinement common to all graphical fans. The idea is to refine the graphical fan $\mathcal{G}_G$ by the usual braid fan $\mathcal{B}_n$. However, $\mathcal{B}_n$ itself is not truly simplicial as it contains a 1-dimensional lineality space. Explicitly, this refining fan is obtained from the braid fan $\mathcal{B}_V$ by cutting each region into two simplices as follows. Associate to any subset $U \subseteq V$ the vector $\boldsymbol{\iota}_U := \sum_{u \in U} \boldsymbol{e}_u - \sum_{v \notin U} \boldsymbol{e}_v$ and consider the fan $\widehat{\mathcal{B}}_V$ whose maximal cells are $\mathsf{C}_\sigma^\varnothing := \operatorname{cone}\{\boldsymbol{\iota}_U \mid U \subsetneq V \text{ upper set of } \sigma\}$ and $\mathsf{C}_\sigma^V := \operatorname{cone}\{\boldsymbol{\iota}_U \mid \varnothing \neq U \subseteq V \text{ upper set of } \sigma\}$ for every total order σ of V.

The fan $\widehat{\mathcal{B}}_V$ is an essential complete simplicial fan in $\mathbb{R}^V$ supported on the $2^{|V|}$ vectors $\boldsymbol{\iota}_U$ for $U \subseteq V$. It has two types of pairs of adjacent maximal cones:

- the pairs $\{\mathsf{C}_\sigma^\varnothing, \mathsf{C}_\sigma^V\}$ for any σ, which yields the linear dependence $\boldsymbol{\iota}_\varnothing + \boldsymbol{\iota}_V = \boldsymbol{0}$,
- the pairs $\{\mathsf{C}_\sigma^X, \mathsf{C}_{\sigma'}^X\}$ for any $X \in \{\varnothing, V\}$ and any total orders $\sigma = PuvS$ and $\sigma' = PvuS$ that differ in the transposition of two consecutive elements. The two rays that are not shared by C_σ^X and $\mathsf{C}_{\sigma'}^X$ are $\boldsymbol{\iota}_{S \cup \{u\}}$ and $\boldsymbol{\iota}_{S \cup \{v\}}$, and the unique linear relation supported on the rays of $\mathsf{C}_\sigma^X \cup \mathsf{C}_{\sigma'}^X$ is given by $\boldsymbol{\iota}_{S \cup \{u\}} + \boldsymbol{\iota}_{S \cup \{v\}} = \boldsymbol{\iota}_S + \boldsymbol{\iota}_{S \cup \{u,v\}}$.

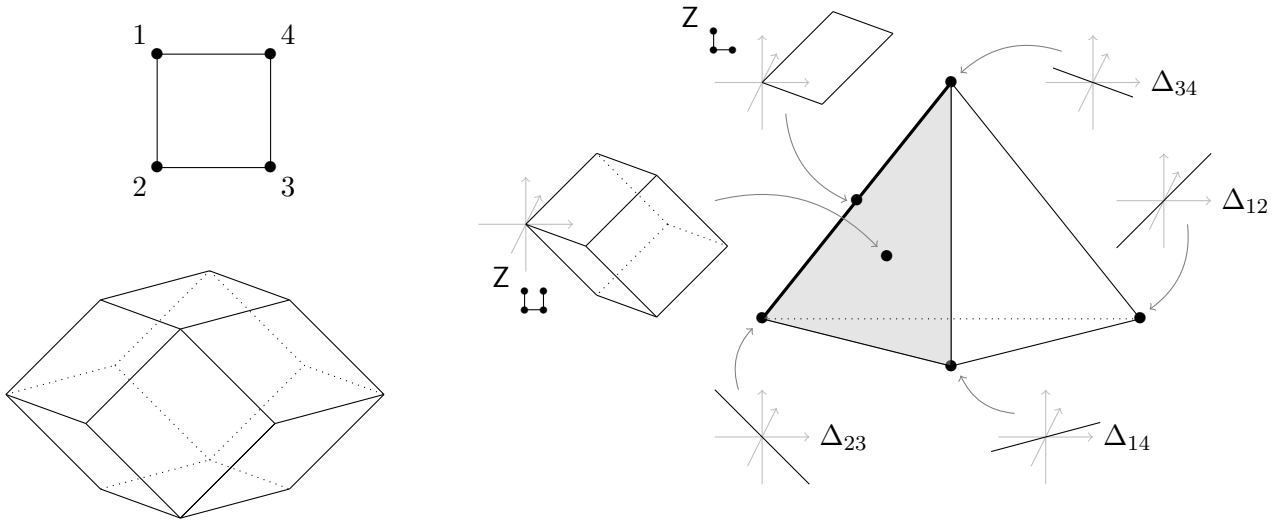


Figure 1: Deformation cone of the graphical zonotope of the 4-cycle. **Left:** 4-cycle G and Z_G. **Right:** Deformation cone $\mathbb{DC}(\mathsf{Z}_G)$. Interior points correspond to polytopes normally equivalent to Z_G.

Moreover, the fan $\widehat{\mathcal{B}}_V$ refines any graphical fan $\mathcal{G}_G$: for any acyclic orientation ω of G, any total order σ on V and any $X \in \{\varnothing, V\}$, we have $\mathsf{C}^X_\sigma \subseteq \mathsf{C}_\omega$ if and only if σ is a linear extension of ω. Applying Proposition 1, we obtain the following description of $\mathbb{DC}(\mathsf{Z}_G)$.

Corollary 2. *The deformation cone $\mathbb{DC}(\mathsf{Z}_G)$ of the graphical zonotope Z_G is the set of polytopes $\left\{\boldsymbol{x} \in \mathbb{R}^V \;\middle|\; \sum_{u\in U} \boldsymbol{x}_u - \sum_{v\notin U} \boldsymbol{x}_v \le \boldsymbol{h}_U \text{ for all } U \subseteq V\right\}$ for all $\boldsymbol{h}$ in the cone of $\mathbb{R}^{2^V}$ defined by the following (possibly redundant) description:*

- $\boldsymbol{h}_\varnothing = -\boldsymbol{h}_V$,
- $\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S\cup\{v\}} = \boldsymbol{h}_S + \boldsymbol{h}_{S\cup\{u,v\}}$ *for each* $\{u,v\} \notin E$ *and* $S \subseteq V \setminus \{u,v\}$,
- $\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S\cup\{v\}} \ge \boldsymbol{h}_S + \boldsymbol{h}_{S\cup\{u,v\}}$ *for each* $\{u,v\} \in E$ *and* $S \subseteq V \setminus \{u,v\}$.

Irredundant description The description of the deformation cone of Corollary 2 is highly redundant, both in the equations describing its linear span and in the inequalities describing its facets. Choosing a basis for the equations and discarding the irredundant inequalities, we obtain the following description, whose proof is the purpose of [16]. We denote by $N(v) \coloneqq \{u \in V \mid \{u,v\} \in E\}$ the neighbors of a vertex v in G.

Theorem 3. *The deformation cone $\mathbb{DC}(\mathsf{Z}_G)$ of the graphical zonotope Z_G is the set of polytopes $\left\{\boldsymbol{x} \in \mathbb{R}^V \;\middle|\; \sum_{u\in U} \boldsymbol{x}_u - \sum_{v\notin U} \boldsymbol{x}_v \le \boldsymbol{h}_U \text{ for all } U \subseteq V\right\}$ for all $\boldsymbol{h}$ in the cone of $\mathbb{R}^{2^V}$ defined by the following irredundant facet description:*

- $\boldsymbol{h}_\varnothing = -\boldsymbol{h}_V$,
- $\boldsymbol{h}_{S\setminus\{u\}} + \boldsymbol{h}_{S\setminus\{v\}} = \boldsymbol{h}_S + \boldsymbol{h}_{S\setminus\{u,v\}}$ *for each* $\varnothing \neq S \subseteq V$ *and any*[4] $\{u,v\} \in \binom{S}{2} \setminus E$,
- $\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S\cup\{v\}} \ge \boldsymbol{h}_S + \boldsymbol{h}_{S\cup\{u,v\}}$ *for each* $\{u,v\} \in E$ *and* $S \subseteq N(u) \cap N(v)$.

Example 4. When G is the 4-cycle, $\mathbb{DC}(\mathsf{Z}_G)$ lives in dimension 16. It is the 4-dimensional simplicial cone with 4 dimensions of lineality, given by 8 equalities and 4 inequalities:

[4]For any non-clique S, only one missing edge is choosen (*e.g.* the lexicographically smallest).

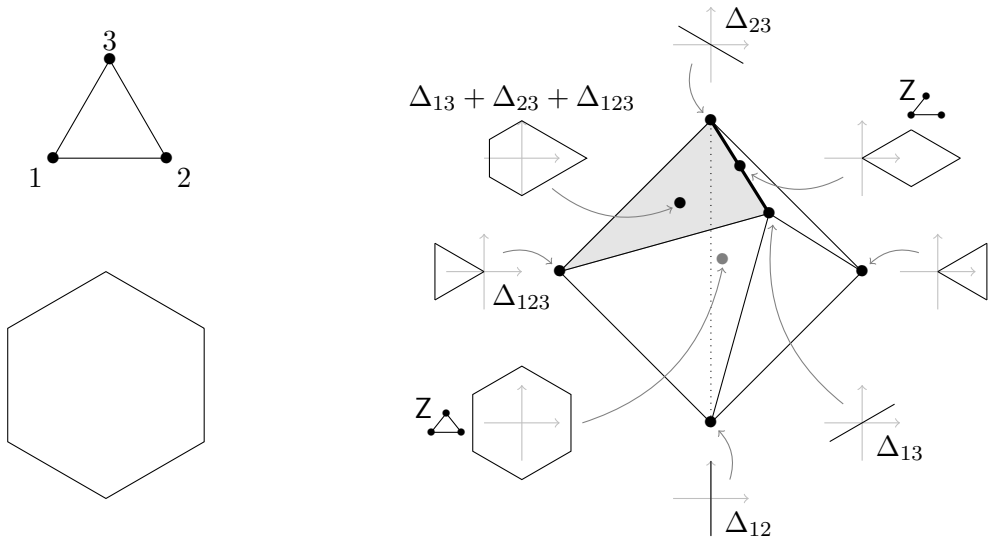


Figure 2: Deformation cone of the graphical zonotope of the 3-cycle. **Left:** 3-cycle G and Z_G (permutahedron of dimension 2). **Right:** Deformation cone $\mathbb{DC}(\mathsf{Z}_G)$, adapted from [17, Figure 16].

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
\boldsymbol{h}_\varnothing & = & -\boldsymbol{h}_{1234} & \boldsymbol{h}_{12}+\boldsymbol{h}_{14} & = & \boldsymbol{h}_{124}+\boldsymbol{h}_1 & \boldsymbol{h}_1+\boldsymbol{h}_2 & \geq & \boldsymbol{h}_{12}+\boldsymbol{h}_\varnothing \\
\boldsymbol{h}_1+\boldsymbol{h}_3 & = & \boldsymbol{h}_{13}+\boldsymbol{h}_\varnothing & \boldsymbol{h}_{12}+\boldsymbol{h}_{23} & = & \boldsymbol{h}_{123}+\boldsymbol{h}_2 & \boldsymbol{h}_2+\boldsymbol{h}_3 & \geq & \boldsymbol{h}_{23}+\boldsymbol{h}_\varnothing \\
\boldsymbol{h}_2+\boldsymbol{h}_4 & = & \boldsymbol{h}_{24}+\boldsymbol{h}_\varnothing & \boldsymbol{h}_{23}+\boldsymbol{h}_{34} & = & \boldsymbol{h}_{234}+\boldsymbol{h}_3 & \boldsymbol{h}_3+\boldsymbol{h}_4 & \geq & \boldsymbol{h}_{34}+\boldsymbol{h}_\varnothing \\
\boldsymbol{h}_{123}+\boldsymbol{h}_{134} & = & \boldsymbol{h}_{1234}+\boldsymbol{h}_{13} & \boldsymbol{h}_{14}+\boldsymbol{h}_{34} & = & \boldsymbol{h}_{134}+\boldsymbol{h}_4 & \boldsymbol{h}_1+\boldsymbol{h}_4 & \geq & \boldsymbol{h}_{14}+\boldsymbol{h}_\varnothing
\end{array}$$

Intersecting $\mathbb{DC}(\mathsf{Z}_G)$ with a hyperplan, we get the 3-simplex illustrated on Figure 1.

Remark 5. *When G is complete, Z_G is a permutahedron and $\mathbb{DC}(\mathsf{Z}_G)$ is the submodular cone given by the irredundant inequalities $\boldsymbol{h}_{S\cup\{u\}}+\boldsymbol{h}_{S\cup\{v\}} \geq \boldsymbol{h}_S+\boldsymbol{h}_{S\cup\{u,v\}}$ for each $\{u,v\}\subseteq S\subset V$. (The usual presentation imposes $\boldsymbol{h}_\varnothing=0$, but both presentations are clearly equivalent up to translation). For $G=K_3$, $\mathbb{DC}(\mathsf{Z}_G)$ is a 4-dimensional cone, intersecting it with an hyperplan we get the bipyramid illustrated on Figure 2. Among the 5 rays of $\mathbb{DC}(\mathsf{Z}_G)$ (i.e. vertices of the bipyramid), 4 of them correspond to Z_H for H a subgraph of K_3, but not the last ray.*

Corollary 6.[5] *The faces $\triangle_K \coloneqq \operatorname{conv}\{\boldsymbol{e}_v \mid v\in K\}$ of the standard simplex $\triangle_V$ corresponding to the non-empty induced cliques K of G form a linear basis of the space spanned by $\mathbb{DC}(\mathsf{Z}_G)$.*

Corollary 7. *The dimension of $\mathbb{DC}(\mathsf{Z}_G)$ is the number of induced cliques in G, the dimension of the lineality space of $\mathbb{DC}(\mathsf{Z}_G)$ is $|V|$, and the number of facets of $\mathbb{DC}(\mathsf{Z}_G)$ is the number of triplets (u,v,S) with $\{u,v\}\in E$ and $S\subseteq N(u)\cap N(v)$.*

Example 8. If G is triangle-free, $\mathbb{DC}(\mathsf{Z}_G)$ has dimension $|V|+|E|$ and $|E|$ facets. If $G=K_V$ is complete, $\mathbb{DC}(\mathsf{Z}_{K_V})$ has dimension $2^{|V|}-1$ and $\binom{|V|}{2}2^{|V|-2}$ facets.

Corollary 9. *The deformation cone $\mathbb{DC}(\mathsf{Z}_G)$ is simplicial (modulo its lineality) if and only if G is triangle-free. In that case, every deformation of Z_G is a zonotope, which is the graphical zonotope of a subgraph of G up to rescaling of the generators.*

Acknowledgements

We are grateful to Raman Sanyal for letting us know about his proof of Corollary 6 with Josephine Yu, as well as for many helpful discussions on the topic, especially on the link between deformation cone and polytope algebra.

[5] This fact was proved independently by Raman Sanyal and Josephine Yu (personal communication).

References

[1] M. Aguiar and F. Ardila. *Hopf monoids and generalized permutahedra*. To appear in *Mem. Amer. Math. Soc.* 2017. arXiv: 1709.07504.

[2] F. Ardila, C. Benedetti, and J. Doker. "Matroid polytopes and their volumes". In: *Discrete Comput. Geom.* 43.4 (2010), pp. 841–854.

[3] F. Chapoton, S. Fomin, and A. Zelevinsky. "Polytopal Realizations of Generalized Associahedra". In: *Canad. Math. Bull.* 45.4 (2002), pp. 537–566.

[4] D. A. Cox, J. B. Little, and H. K. Schenck. *Toric varieties*. Vol. 124. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011, pp. xxiv+841.

[5] V. I. Danilov and G. A. Koshevoy. "Cores of cooperative games, superdifferentials of functions, and the Minkowski difference of sets". In: *J. Math. Anal. Appl.* 247.1 (2000), pp. 1–14.

[6] J. A. De Loera, J. Rambau, and F. Santos. *Triangulations: Structures for Algorithms and Applications*. Vol. 25. Algorithms and Computation in Math. Springer, 2010.

[7] J. Edmonds. "Submodular functions, matroids, and certain polyhedra". In: *Combinatorial Structures and their Applications*. Gordon and Breach, New York, 1970, pp. 69–87.

[8] S. Fujishige. *Submodular functions and optimization*. Second. Vol. 58. Annals of Discrete Mathematics. Elsevier B. V., Amsterdam, 2005, pp. xiv+395.

[9] M. Joswig, M. Klimm, and S. Spitz. "Generalized permutahedra and optimal auctions". 2021. arXiv: 2108.00979.

[10] S. Kim. "Shellable complexes and topology of diagonal arrangements". In: *Discrete Comput. Geom.* 40.2 (2008), pp. 190–213.

[11] P. McMullen. "Representations of polytopes and polyhedral sets". In: *Geometriae Dedicata* 2 (1973), pp. 83–99.

[12] P. McMullen. "Indecomposable convex polytopes". In: *Israel J. Math.* 58.3 (1987), pp. 321–323.

[13] W. Meyer. "Indecomposable polytopes". In: *Trans. Amer. Math. Soc.* 190 (1974), pp. 77–86.

[14] F. Mohammadi, C. Uhler, C. Wang, and J. Yu. "Generalized permutohedra from probabilistic graphical models". In: *SIAM J. Discrete Math.* 32.1 (2018), pp. 64–93.

[15] J. Morton, L. Pachter, A. Shiu, B. Sturmfels, and O. Wienand. "Convex rank tests and semigraphoids". In: *SIAM J. Discrete Math.* 23.3 (2009), pp. 1117–1134.

[16] A. Padrol, V. Pilaud, and G. Poullot. "Deformed graphical zonotopes". 2021. arXiv: 2111.12422.

[17] A. Padrol, V. Pilaud, and J. Ritter. "Shard polytopes". In: (2020). Preprint. DOI: 10.48550/ARXIV.2007.01008.

[18] V. Pilaud and F. Santos. "Quotientopes". In: *Bull. Lond. Math. Soc.* 51.3 (2019), pp. 406–420.

[19] A. Postnikov. "Permutohedra, associahedra, and beyond". In: *Int. Math. Res. Not. IMRN* 6 (2009), pp. 1026–1106.

[20] A. Postnikov, V. Reiner, and L. K. Williams. "Faces of generalized permutohedra". In: *Doc. Math.* 13 (2008), pp. 207–273.

[21] K. Przesławski and D. Yost. "More indecomposable polyhedra". In: *Extracta Math.* 31.2 (2016), pp. 169–188.

[22] P. H. Schoute. *Analytical treatment of the polytopes regularly derived from the regular polytopes. Section I: The simplex.* Vol. 11. 3. 1911, 82 s.

[23] G. C. Shephard. "Decomposable convex polyhedra". In: *Mathematika* 10 (1963), pp. 89–95.

[24] R. P. Stanley. "An introduction to hyperplane arrangements". In: *Geometric combinatorics*. Vol. 13. IAS/Park City Math. Ser. Amer. Math. Soc., Providence, RI, 2007, pp. 389–496.

[25] R. P. Stanley. *Enumerative combinatorics. Volume 1*. Second. Vol. 49. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2012, pp. xiv+626.

New Ramsey multiplicity bounds and search heuristics

Olaf Parczyk[1*], Sebastian Pokutta[2†,‡], Christoph Spiegel[3†], Tibor Szabó[4*]

[*]Freie Universität Berlin, Institute of Mathematics, Berlin, Germany
[†]Zuse Institute Berlin, Department for AI in Society, Science, and Technology, Berlin, Germany
[‡]Technische Universität Berlin, Institute of Mathematics, Berlin, Germany

The full version of this work can be found in [21] and will be published elsewhere. This research was partially funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – The Berlin Mathematics Research Center MATH+ (EXC-2046/1, project ID: 390685689).

Abstract

We study two related problems concerning the number of monochromatic cliques in two-colorings of the complete graph that go back to questions of Erdős. Most notably, we improve the 25-year-old upper bound of the K_4 and K_5 Ramsey multiplicity due to Thomason and settle the minimum number of independent sets of size 4 in graphs with clique number at most 4. We also introduce a notion of off-diagonal Ramsey multiplicity, obtaining two tight results when counting triangles in one color, and more broadly motivate this question by relating it to the study of which pairs of clique and independent set densities can be realized as the limit of some sequence of graphs.

1 New Ramsey multiplicity bounds

We denote by $k_t(G)$ the number of cliques on t vertices in a graph G and let $k_t(n) = \min\{k_t(G)+k_t(\overline{G}) : |G| = n\}$ be the minimum number of monochromatic copies of K_t in a 2-edge-coloring of the complete graph on n vertices. We know by Ramsey's theorem that $k_t(n) > 0$ when n is sufficiently large depending on t. The study of $k_t(n)$ or the limit

$$c_t = \lim_{n\to\infty} k_t(n)/\binom{n}{t}$$

is known as the Ramsey multiplicity problem and has received a fair amount of attention in the past. Note that the limit exists as $k_t(n)$ is increasing with n. A result of Goodman [9] precisely establishes $k_3(n)$ for any $n \in \mathbb{N}$, implying that asymptotically one cannot do better than a random coloring since $c_3 = 1/4$. This lead Erdős to conjecture [4] that this observation holds for arbitrary t, that is that $c_t = 2^{1-\binom{t}{2}}$. This was soundly rejected by Thomason [31] for any $t \geq 4$ and the best current upper bounds for $t \in \{4, 5\}$ of $c_4 < 0.03029$ and $c_5 < 0.001720$ are also due to Thomason [32]. Here we present the following improved upper bounds for c_4 and c_5.

Theorem 1. *We have* $c_4 < 0.03015$ *and* $c_5 < 0.001708$.

The bounds are established through constructions that were found using computer search heuristics [13, 8] that have previously proven effective in establishing lower bounds for Ramsey numbers [5, 6, 17]. Given

[1]Email: parczyk@mi.fu-berlin.de
[2]Email: pokutta@zib.de
[3]Email: spiegel@zib.de
[4]Email: szabo@mi.fu-berlin.de

a recent surge in interest in obtaining combinatorially relevant constructions through computational means [14, 28, 34, 35], we think these methods have so far been been somewhat overlooked as an option to go beyond the point where exhaustive searches are feasible.

We ran heuristic searches to construct graphs on up to 40 vertices and the smallest graph we found whose blow-up sequence establishes a value below 1/32 was of order 33, giving a value of 0.03118. The smallest previously known such graph was described by Thomason [31] and was of order 36. We also found a graph on 32 vertices whose *weighted* blow-up sequence gives a value slightly below 1/32. Given that the blow-up sequences of small graphs seem to yield little of interest, we considered Cayley graphs next. The best previous construction for c_4 was given by the blow-up sequence of a Cayley graph of in $C_3^{\times 2} \times C_2^{\times 5}$ (order 288) and a search in that particular group yielded no improvement. However, already in $C_3 \times C_2^{\times 6}$ (order 192) we were able to find a graph whose blow-up sequence improves upon the value found by Thomason, giving an upper bound of 0.03027. Going up to $C_3 \times C_2^{\times 8}$ (order 768) produced the graph whose blow-up sequence gives the upper bound stated in Theorem 1. Regarding the value for c_5, the previous best construction can be described as the blow-up sequence of a Cayley graph in $C_3 \times C_2^{\times 6}$ (order 192). A search run on Cayley graphs in this group yielded the slight improvement presented in Theorem 1.

We note that there seems to be no particular significance to the fact that we derived these constructions in groups defined only through direct products. In fact, running the search for c_4 on all groups of order at most 192 revealed (1) that in general good constructions seem to be found in groups of order $2 \cdot 3^n$ and (2) many groups of that order perform significantly better than the group $C_3 \times C_2^{\times n}$. For groups of order 192 for example the best value we found was around 0.03021. Unfortunately we were unable to determine any patterns indicating which groups might be preferable when going to groups of order 384 or 768. The sheer number of these groups and the amount of cliques to consider makes it impossible to run a search on anything more than a small selection of them.

Lower bounds have also been established for these problems. Giraud [7, 36] proved that $c_4 \geq 1/46$ using traditional means, though lately answers to these kinds of questions have largely been achieved computationally using the tool of Flag Algebras due to Razborov [24, 26]. Essentially this approach allows one to minimize linear combinations of subgraph densities by solving a semi-definite program. Niess [18] and Sperfeld [30] independently used this approach to establish a lower bound for c_4 of around $1/35 > 0.028571$ and Grzesik et al. [10] later improved that to 0.0296. To the best of our knowledge, no formal lower bound has been published for c_5. Using the Flag Algebra approach, we obtained the lower bound

$$c_5 \geq 0.001524. \tag{1}$$

Note that for the general case Conlon [2] provides an asymptotic lower bound of $c_t \geq 2.18^{-t^2(1+o(1))}$.

A central question underlying the Ramsey multiplicity problem is whether or not a tight upper bound can be attained by the sequence of (possibly weighted) sequence of blow-ups of some finite sized graph. For c_3 this holds true as the bound of Goodman is attained by, among other constructions, the sequence of blow-ups of K_2. Already for c_4 this question remains unanswered and our results do not give a clear indication either way, other than suggesting that any such finite construction might need to be truly massive. We are however able to answer this question in the affirmative for some specific cases of other problems that are closely linked to the Ramsey multiplicity problem.

2 Independent set density for graphs with bounded clique number

A related question likewise raised by Erdős [4] concerns the minimum number $k'_{s,t}(n) = \min\{k_s(\overline{G}) : |G| = n, k_t(G) = 0\}$ of cliques of order s in the complement of a graph of order n with clique number bounded by $t-1$. The associated limit is defined as

$$g_{s,t} = \lim_{n\to\infty} k'_{s,t}(n) / \binom{n}{s}.$$

Note that obviously $g_{t,t} \geq c_t$ for any $t \geq 2$. We trivially have $g_{s,2} = 1$ and the fact that $g_{2,t} = 1/(t-1)$ is established by Turán's theorem [33]. Erdős [4] asked if the upper bound given by the Turán graphs $T_{t-1}(n)$ as $n \to \infty$ is tight in general, that is if $g_{s,t} = (t-1)^{1-s}$. This holds for triangles, that is when $s = t = 3$, as an easy consequence of the previously mentioned result of Goodman [9]. Nikiforov [19] however showed that this upper bound can be sharp only for a finite number of pairs $s, t \geq 3$. Das et al. [3] as well as Pikhurko and Vaughan [23] established tight values for $g_{3,t}$ and $g_{s,3}$ when $4 \leq s, t \leq 7$, confirming Erdős' intuition for the former and disproving it for the latter. Pikhurko and Vaughan [23] also found a construction based on weighted blow-ups of a $(3,4)$-Ramsey graph of order 8 bounding $g_{4,4}$ away from the value given by $T_3(n)$ that is conjectured to be tight. We present one further tight value for $g_{s,t}$. Let $C_{R(3,5)}$ denote the unique $(3,5)$-Ramsey graph of order 13, that is the Cayley graph on $\mathbb{Z}_{13}$ whose edge relations are given by the cubic-non-residues.

Theorem 2. *We have $g_{4,5} = 29 \cdot 13^{-3}$ and the problem is perfectly $C_{R(3,5)}$-stable.*

The notion of perfect stability was introduced in [22] and strengthens the standard notion of stability. The upper bound is given by the sequence of uniform blow-ups of $C_{R(3,5)}$. The lower bound matching that construction as well as the stability results was established using the Flag Algebra approach. The proof of stability largely follows the template laid out by Pikhurko et al. [22] for these types of statements, but does require some problem specific ad-hoc arguments.

The fact that Ramsey graphs are a good source of constructions for this problem was previously already noted by Nikiforov [19] and Das et al. [3]. We found several more such graphs whose sequence of (sometimes weighted) blow-ups give good upper bounds for additional values of $g_{s,t}$, but were unable to establish matching lower bounds. The most reasonable open conjecture out of all studied values seems to be that $g_{5,5} = 61 \cdot 13^{-4}$, where the upper bound also comes from $C_{R(3,5)}$. We also remark that more general bounds for $g_{s,t}$ were given by Nikiforov [19] and Sawin [29], who studied the close connection to (multicolor) Ramsey numbers.

3 Off-diagonal Ramsey multiplicity

Given the slow progress to determine the minimum number of monochromatic cliques of size 4, that is determining c_4, while counting monochromatic triangles is easy, we think it is of interest if to see if the problem becomes more tangible when considering triangles in one color and cliques of size 4 in the other. We in fact obtain a precise answer for this problem, where a tight bound is again obtained through the blow-up of a finite sized graph. More broadly, given that off-diagonal variants of Ramsey problems often offer interesting insights and that the parameter $g_{s,t}$ from the previous section has been studied for arbitrary pairs $s, t \geq 2$, we think that it is natural to consider determining the following off-diagonal version of the Ramsey multiplicity problem:

$$c_{s,t} = \lim_{n\to\infty} \min \left\{ \frac{k_s(\overline{G})}{\binom{n}{s}} + \frac{k_t(G)}{\binom{n}{t}} : |G| = n \right\}. \tag{2}$$

Observe that clearly $c_{t,t} = c_t$ as well as $c_{s,t} \leq \min\{g_{s,t}, g_{t,s}\}$. From a result of Reiher [27] it follows that $c_{2,t} = g_{2,t}$ for every $t \geq 3$. Here we establish the first exact results when $s, t \geq 3$ and $s \neq t$. Let C_S denote the Schläfli graph on 27 vertices.

Theorem 3. *We have $c_{3,4} = 689 \cdot 3^{-8}$ and the problem is perfectly C_S-stable. We also have $c_{3,5} = 24011 \cdot 3^{-12}$ as well as $0.007688 < c_{4,5} \leq 0.007932$.*

Note that $c_{s,t} < g_{s,t}$ for each of these values of s and t. The upper bound for $c_{3,4}$ is given by the sequence of blow-ups of the Schläfli graph. The upper bound for $c_{3,5}$ is more easily described as an upper bound of $c_{5,3}$, in which case it is given by the sequence of the blow-ups of the complement of the Schläfli graph. The upper bound for $c_{4,5}$ is given by the sequence of blow-ups of a vertex-transitive graph on 128 vertices. Lower bounds and stability are again established using the Flag Algebra approach.

We note that there is no particular reason for our choice to weight the contribution from both summands equally in $c_{s,t}$ and other choice would also be reasonable, for example the weighting given by linearly connecting the points indicated by $g_{s,t}$ and $g_{t,s}$. In fact, the previously introduced problems are are part of a broader question in which one would like to understand which pairs of clique and independent set densities can be realized as the limit of some sequence of graphs.

4 The full tradeoff between cliques and independent sets

The study of $c_{s,t}$ and $g_{s,t}$ are part of a broader question in which one would like to understand the full tradeoff between the number of cliques of size t and independent sets of size s in a graph. The goal is to characterise the region $\Omega_{s,t} \subseteq [0,1]^2$ of pairs of clique and independent set densities that can occur in the limit of a sequence of graphs. We say that a tuple $(x,y) \in [0,1]^2$ is realised by a sequence of graphs $(G_n)_{n\in\mathbb{N}}$ of growing order, where without loss of generality G_n is of order n, if $\lim_{n\to\infty} k_s(\overline{G_n})/\binom{n}{s} = x$ and $\lim_{n\to\infty} k_t(G_n)/\binom{n}{t} = y$ and formally define $\Omega_{s,t} \subseteq [0,1]^2$ to be the set of all tuples that are realised by some sequence of graphs. For $s = 2$ and with K_t replaced by any quantum graph, that is an arbitrary linear combination of graphs, this set was already systematically studied by Liu, Mubayi, and Reiher [15]. Similar to them, let us define

$$c_{s,t}(x) = \inf\{y\colon (x,y) \in \Omega_{s,t}\} \quad \text{and} \quad C_{s,t}(x) = \sup\{y\colon (x,y) \in \Omega_{s,t}\}.$$

We can show that $\Omega_{s,t}$ behaves nicely for any $s,t \geq 2$, that is it is completely characterised by its lower and upper bounding curves $c_{s,t}(x)$ and $C_{s,t}(x)$.

Proposition 4. *$\Omega_{s,t}$ is compact and defines a simply connected region for any $s,t \geq 2$. The curves $c_{s,t}(x)$ and $C_{s,t}(x)$ are decreasing, continuous, and almost everywhere differentiable for any $s,t \geq 2$.*

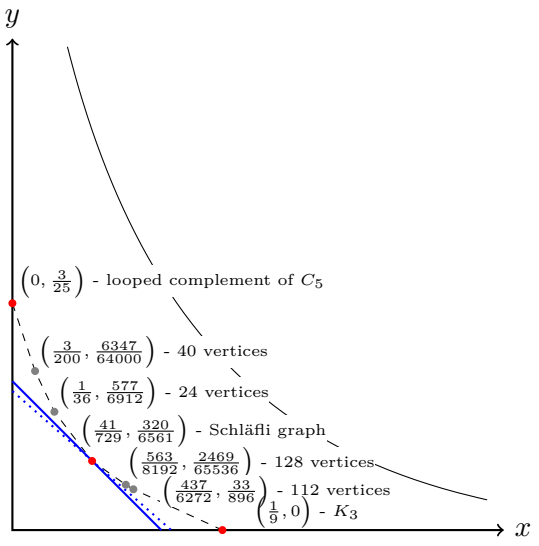


Figure 1: The known bounds on the region $\Omega_{3,4}$. The blue line is the lower bound $c_{3,4} \geq 689 \cdot 3^{-8}$, the blue dotted line is an additional linear lower bound for $c_{3,4}(x)$, red dots represent optimal constructions, and grey dots represent additional ones.

$C_{s,t}(x)$ is the easier of the two to establish and is precisely given by the Kruskal-Katona theorem when $\min\{s,t\} = 2$. It was more generally determined for arbitrary $s,t \geq 2$ by Huang et al. [12]. On the other hand, much less is known about $c_{s,t}(x)$. Clearly $c_{s,t}(0) = g_{s,t}$ and $c_{s,t}(x) = 0$ if and only if $x \geq g_{t,s}$, that is $(0, g_{s,t})$ and $(g_{t,s}, 0)$ are the points where the curve $c_{s,t}(x)$ intersects the axes when $0 \leq x \leq g_{t,s}$. Moreover, $c_{s,t} = \min_x c_{s,t}(x) + x$ and therefore $c_{s,t}(x) \geq c_{s,t} - x$.

For $s = 2$ one is interested in the minimum possible density of cliques of size t in a graph of given edge density and in this case $c_{2,t}(x)$ was completely determined; Razborov [25] gave an answer for $t = 3$, Nikiforov [20] for $t = 4$, and Reiher [27] for arbitrary t. The region $\Omega_{3,3}$ on the other hand was established by Huang et al. [11]. Beyond that very little is known about the shape of $c_{s,t}(x)$, providing some additional motivation for studying the parameters $c_{s,t}$ and $g_{s,t}$ stated in the introduction.

Theorems 1 and 3 of course imply lower bounds of $c_{s,t}(x)$ that are linear in x for some specific values of s and t. Let us take a closer look at the smallest open case, that is $s = 3$ and $t = 4$. Calculating the $\overline{K_3}$ and K_4 density of the sequence of blow-ups of the Schläfli graph, we get that $c_{3,4}(41 \cdot 3^{-6}) = 320 \cdot 3^{-8}$,

establishing one precise value of the curve besides the ones on the axes. Additionally, we can show that $c_{3,4}(x)$ is not differentiable at this point by establishing a second tight lower bound using a differently weighted version of $c_{3,4}$.

Noting that by Das et al. [3] and Pikhurko and Vaughan [23] the value of $g_{3,4} = c_{3,4}(0) = c_{4,3}(1/9)$ is determined by the sequence of blow-ups of K_3 and the value of $g_{4,3} = c_{3,4}(3/25) = c_{4,3}(0)$ by that of the looped complement of C_5, it also seems reasonable to ask if the Schläfli graph could mark a first 'extremal point' of the curve $c_{3,4}(x)$ that does not lie on either axis, the same way that K_t does for $c_{2,t}(x)$, or if alternatively the blow-up sequence of another vertex-transitive graph on fewer than 27 vertices marks such a point.

Somewhat surprisingly, neither of these two options seems to be true: there is no sequence of blow-ups of a vertex-transitive graph on up to 47 vertices that determines a point in convex position with the points given by the Schläfli graph and K_3, where one might expect points of discontinuity to be more easily described. There are however various blow-up sequences of vertex transitive graphs on at least 112 vertices that are in convex position with those points. On the other hand, between the points given by the Schläfli graph and the looped complement of C_5, where we would expect constructions to become increasingly complex, we first find a vertex-transitive graph on only 24 vertices determining a point in convex position with the two other points. Lest one assume that this might indicate a pattern, there also exists a vertex-transitive graph on 40 vertices that determines a point in convex position with this point and the one given by the looped complement of C_5. Our results are illustrated in Figure 1.

5 Parallels to problems in Additive Combinatorics

We note that similar problems have been studied in Additive Combinatorics, where one is interested in minimizing the number of monochromatic solutions to some system of linear equations. Cameron, Cilleruelo and Serra [1] showed that in finite groups the number of monochromatic solutions to any equation in an odd number of variables is minimized by the random coloring. Wolf [36] as well as Lu and Peng [16] studied the question of how many 4-term arithmetic progressions a two coloring of $\mathbb{Z}_n$ can contain. Interestingly, the upper bounds given for this type of problem likewise consist of a type of 'blow-up' of a finite construction, making the search heuristics used here also applicable in this context. Denoting the minimum fraction of monochromatic k-term arithmetic progressions in a 2-coloring of $\mathbb{Z}_n$ by $m_k(\mathbb{Z}_n)$, it is easy to derive the following lemma through a repetion of a partial coloring.

Lemma 5 (Lemma 4.1 and Theorem 1.5 in [16])**.** *Let $k \geq 3$ be an integer and A be a partial coloring of $\mathbb{Z}_m$, where ℓ distinct elements in arithmetic progression are uncolored for some ℓ dividing m. If $m_k(A) = 1/n$, we get*

$$\liminf_{n\to\infty} m_k(\mathbb{Z}_n) \leq \frac{1}{n+\ell}, \tag{3}$$

where $m_k(A)$ denotes the maximum fraction of monochromatic k-term arithmetic progressions for any coloring of the ℓ remaining elements in A.

Lu and Peng [16] found a construction for $k = 4$ with $n = 11$ and $\ell = 1$ giving $\liminf m_4(\mathbb{Z}_n) \leq 1/12$ and conjecture this to be tight. Using a computer search, we found partial colorings of $\mathbb{Z}_{44}$ and $\mathbb{Z}_{226}$, that establish

$$\liminf m_5(\mathbb{Z}_n) \leq 1/48 \quad \text{and} \quad \liminf m_6(\mathbb{Z}_n) \leq 1/228, \tag{4}$$

with the former improving upon a bound found by of Lu and Peng [16]. We believe that our constructions for $k = 5, 6$ provide tight bounds as well.

References

[1] P. J. Cameron, J. Cilleruelo, and O. Serra. "On monochromatic solutions of equations in groups". In: *Revista Matemática Iberoamericana* 23.1 (2007), pp. 385–395.

[2] D. Conlon. "On the Ramsey multiplicity of complete graphs". In: *Combinatorica* 32.2 (2012), pp. 171–186. DOI: 10.1007/s00493-012-2465-x.

[3] S. Das, H. Huang, J. Ma, H. Naves, and B. Sudakov. "A problem of Erdős on the minimum number of k-cliques". In: *Journal of Combinatorial Theory, Series B* 103.3 (2013), pp. 344–373.

[4] P. Erdős. "On the number of complete subgraphs contained in certain graphs". In: *Magyar Tud. Akad. Mat. Kutató Int. Közl* 7.3 (1962), pp. 459–464.

[5] G. Exoo. "Some new Ramsey colorings". In: *the electronic journal of combinatorics* (1998), R29–R29.

[6] G. Exoo and M. Tatarevic. "New Lower Bounds for 28 Classical Ramsey Numbers". In: *The Electronic Journal of Combinatorics* (2015), P3–11.

[7] G. Giraud. "Sur le probleme de Goodman pour les quadrangles et la majoration des nombres de Ramsey". In: *Journal of Combinatorial Theory, Series B* 27.3 (1979), pp. 237–253.

[8] F. Glover. "Future paths for integer programming and links to artificial intelligence". In: *Computers & operations research* 13.5 (1986), pp. 533–549.

[9] A. W. Goodman. "On sets of acquaintances and strangers at any party". In: *The American Mathematical Monthly* 66.9 (1959), pp. 778–783.

[10] A. Grzesik, J. Lee, B. Lidickỳ, and J. Volec. "On tripartite common graphs". In: *arXiv preprint arXiv:2012.02057* (2020). DOI: 10.48550/ARXIV.2012.02057.

[11] H. Huang, N. Linial, H. Naves, Y. Peled, and B. Sudakov. "On the 3-local profiles of graphs". In: *Journal of Graph Theory* 76.3 (2014), pp. 236–248.

[12] H. Huang, N. Linial, H. Naves, Y. Peled, and B. Sudakov. "On the densities of cliques and independent sets in graphs". In: *Combinatorica* 36.5 (2016), pp. 493–512.

[13] S. Kirkpatrick, C. D. Gelatt Jr, and M. P. Vecchi. "Optimization by Simulated Annealing". In: *science* 220.4598 (1983), pp. 671–680.

[14] B. Lidickỳ and F. Pfender. "Semidefinite programming and Ramsey numbers". In: *arXiv preprint arXiv:1704.03592* (2017). DOI: 10.48550/ARXIV.1704.03592.

[15] X. Liu, D. Mubayi, and C. Reiher. "The feasible region of induced graphs". In: *arXiv preprint arXiv:2106.16203* (2021). DOI: 10.48550/ARXIV.2106.16203.

[16] L. Lu and X. Peng. "Monochromatic 4-term arithmetic progressions in 2-colorings of $\mathbb{Z}_n$". In: *Journal of Combinatorial Theory, Series A* 119.5 (2012), pp. 1048–1065.

[17] B. D. McKay and S. P. Radziszowski. "Subgraph counting identities and Ramsey numbers". In: *journal of combinatorial theory, Series B* 69.2 (1997), pp. 193–209.

[18] S. Nieß. "Counting monochromatic copies of K_4: a new lower bound for the Ramsey multiplicity problem". In: *arXiv:1207.4714* (2012). DOI: 10.48550/ARXIV.1207.4714.

[19] V. Nikiforov. "On the minimum number of k-cliques in graphs with restricted independence number". In: *Combinatorics, Probability and Computing* 10.4 (2001), pp. 361–366.

[20] V. Nikiforov. "The number of cliques in graphs of given order and size". In: *Transactions of the American Mathematical Society* 363.3 (2011), pp. 1599–1618.

[21] O. Parczyk, S. Pokutta, S. Christoph, and T. Szabó. "New Ramsey Multiplicity Bounds and Search Heuristics". In: *arXiv preprint* (2022).

[22] O. Pikhurko, J. Sliačan, and K. Tyros. "Strong forms of stability from flag algebra calculations". In: *Journal of Combinatorial Theory, Series B* 135 (2019), pp. 129–178.

[23] O. Pikhurko and E. R. Vaughan. "Minimum number of k-cliques in graphs with bounded independence number". In: *Combinatorics, Probability and Computing* 22.6 (2013), pp. 910–934.

[24] A. A. Razborov. “Flag algebras”. In: *The Journal of Symbolic Logic* 72.4 (2007), pp. 1239–1282.

[25] A. A. Razborov. “On the minimal density of triangles in graphs”. In: *Combinatorics, Probability and Computing* 17.4 (2008), pp. 603–618.

[26] A. A. Razborov. “On 3-hypergraphs with forbidden 4-vertex configurations”. In: *SIAM Journal on Discrete Mathematics* 24.3 (2010), pp. 946–963.

[27] C. Reiher. “The clique density theorem”. In: *Annals of Mathematics* (2016), pp. 683–707.

[28] F. Rowley. “Improved Lower Bounds for Multicolour Ramsey Numbers using SAT-Solvers”. In: *arXiv preprint arXiv:2203.13476* (2022). DOI: 10.48550/ARXIV.2203.13476.

[29] W. Sawin. “An improved lower bound for multicolor Ramsey numbers and the half-multiplicity Ramsey number problem”. In: *arXiv preprint arXiv:2105.08850* (2021). DOI: 10.48550/ARXIV.2105.08850.

[30] K. Sperfeld. “On the minimal monochromatic K_4-density”. In: *arXiv preprint arXiv:1106.1030* (2011). DOI: 10.48550/ARXIV.1106.1030.

[31] A. Thomason. “A disproof of a conjecture of Erdős in Ramsey theory”. In: *Journal of the London Mathematical Society* 2.2 (1989), pp. 246–255.

[32] A. Thomason. “Graph products and monochromatic multiplicities”. In: *Combinatorica* 17.1 (1997), pp. 125–134.

[33] P. Turán. “On an external problem in graph theory”. In: *Mat. Fiz. Lapok* 48 (1941), pp. 436–452.

[34] A. Z. Wagner. “Refuting conjectures in extremal combinatorics via linear programming”. In: *Journal of Combinatorial Theory, Series A* 169 (2020), p. 105130.

[35] A. Z. Wagner. *Constructions in combinatorics via neural networks*. Preprint. 2021. DOI: 10.48550/ARXIV.2104.14516.

[36] J. Wolf. “The minimum number of monochromatic 4-term progressions in $\mathbb{Z}_p$”. In: *Journal of Combinatorics* 1.1 (2010), pp. 53–68.

Fast positive Plücker trees

Julian Pfeifle[1]

Dept. de Matemàtiques, Universitat Politècnica de Catalunya, Jordi Girona 1-3, Barcelona

Abstract

We report on a re-implementation in C++ of the search for non-realizability certificates for simplicial spheres in the form of positive Plücker trees [16]. On the one hand, this implementation is orders of magnitudes faster than the previous one, does not need linear or integer solvers anymore, and finds certificates in seconds that are completely out of reach of the related implementation by Gouveia et al [9]. On the other, our method misses some certificates that Gouveia et al do find, so that these methods are genuinely complementary.

1 Introduction

Since the early days of discrete and combinatorial geometry, the question of whether a given simplicial sphere Σ has a realization as a convex polytope has been very fruitful. Here, a *simplicial sphere* is a simplicial complex whose underlying space is homeomorphic to a sphere.

Since every 2-dimensional simplicial sphere can be realized as the boundary complex of a convex 3-dimensional polytope, it came as a big surprise when Goodman & Pollack [6] discovered in 1985 that *"There are far fewer polytopes that we thought"* — meaning that asymptotically, there are only $p_d(n) \in 2^{O_d(n \log n)}$ combinatorial types of d-dimensional simplicial polytopes on n vertices. Shortly thereafter, Kalai [10] established the astronomically larger lower bound $s_d(n) \in 2^{\Omega(n^{\lfloor d/2 \rfloor})}$ for the number of combinatorial types of d-dimensional simplicial spheres on n vertices. The best current result on these matters is the improvement to $s_{2k-1} \in 2^{\Omega(n^k)}$ for odd dimension due to Nevo, Santos and Wilson [15], who improved on earlier bounds for 3-spheres by the present author and Ziegler [17].

This problem is genuinely hard: Already for $d = 4$ (in the non-simplicial case), it is NP-hard to decide whether a polytopal complex homeomorphic to a sphere has a convex realization [19], [14], [21]. According to Adiprasito and Padrol [1], the realization problem for neighborly spheres is "universal", i.e., for every "primary basic open semi-algebraic set over $\mathbb{Z}$", there exists a neighborly polytope with that realization space.

2 The algorithmic decision problem

All the mentioned results are purely asymptotic and say nothing about how to decide whether a concrete simplicial d-sphere Σ on n vertices is realizable or not. So how do we go about this?

All known algorithmic approaches to date make use, in one form or another, of the so-called *Grassmann–Plücker relations*

$$\Gamma(I,J) \;=\; \sum_{k=1}^{d+2} (-1)^k [i_1, \ldots, i_d, j_k] [j_1, \ldots, \widehat{j_k}, \ldots, j_{d+2}] \;=\; 0, \tag{1}$$

[1]Email: julian.pfeifle@upc.edu A similar version has been submitted to MEGA2021. This research is supported by the grant PID2019-106188GB-I00 from the Spanish Ministry of Education (MEC)

where $I \in \binom{[n]}{d}$ and $J \in \binom{[n]}{d+2}$ are tuples of indices, the brackets are the so-called *Plücker coordinates* of the simplicial sphere Σ, and the hat notation $\widehat{j_k}$ means omission of the index, as usual. In a realization, the Plücker coordinates can be thought of as the determinants of $(d+1)$-tuples of vertices; see [2, Theorem 1.8], [12, Proposition 2.2.10] and [13, Theorem 14.6].

We can derive a contradiction to realizability from these relations by finding polynomial combinations of them that sum to zero, while in any supposed convex realization that combination cannot vanish. These polynomial combinations were introduced by Bokowski and Sturmfels [2], who called them "final polynomials" because they lead to the demise of the supposed realization. Sturmfels and Richter-Gebert later added linear programming to the toolbox [18], by focusing on the special case of so-called "biquadratic final polynomials" that could be found by solving a linear program.

Much later, Gouveia, Macchia, Brandt and Wiebe studied different models of the space of all possible realizations of a convex polytope [8], and introduced *slack ideals* and the *slack variety* [7], [3], [11]. Their latest paper [9] exploits the positivity of the *slack coordinates* of a polytope, and uses linear programming to find polynomial combinations of these that sum to zero, and therefore provide a final polynomial for the sphere in question. Extensive use of chains of specializations of these slack coordinates helps to keep the size of the linear programs under control.

The present author's earlier contribution to these matters [16] is related to those of Gouveia et al, but differs in several respects. While those authors treat all polynomials completely explicitly as sums of monomials, and allow themselves to specialize certain variables in order to decrease the size of their linear programs, we resolutely stick to Plücker coordinates and their combinations in Grassmann–Plücker relations $\Gamma(I, J)$, which we encode combinatorially via their index sets I and J. Moreover, we only consider combinations of Plücker coordinates that have an underlying tree structure, in which two relations are connected by an edge whenever they share exactly one Plücker coordinate, and that coordinate has different sign in the two relations. Our actual search for the final polynomial was done by solving an integer program on the indicator variables for the vertices and edges of the resulting tree; in this, our approach and that of Gouveia et al are broadly similar.

In terms of the results, on the one hand the implementation in [16] obtains several certificates that are completely out of reach for Gouveia et al in terms of both the size of the resulting certificate, as well as the dimension and number of vertices of the spheres that can be processed. On the other hand, [9] obtains several certificates that our method does not, for instance several of the smaller combinatorial prismatoids of Criado and Santos [4], and also finds slightly smaller certificates in several cases.

3 The new implementation

Our new contribution is the re-implementation in C++, using the `polymake` framework [5], of the algorithm from [16] directly as a combinatorial search. We do not rely on integer or linear programming software anymore, which cuts down quite a lot on execution time and memory.

Example 1. Consider Zheng's balanced neighborly 3-sphere on 16 vertices [22] with 80 facets, which has a non-realizability certificate in form of a tree on six Grassmann–Plücker relations [16, Theorem 5.2]. Computing such a certificate remains out of reach even for the new implementation of Gouveia et al [9, Table 1], while the implementation of [16, Remark 5.4] needs 20 GB of memory and 7.6 hours to construct and solve an integer program with about 7 million variables and 21 million non-zeros.

The new C++ implementation, on the other hand, finds the certificate in about 5 seconds using very modest amounts of memory! ◇

3.1 Outline of the algorithm

At the heart of the algorithm lies the fact that any Grassmann–Plücker coordinate $\sigma = [i_1, \ldots, i_{d+1}]$ can be brought into one of two normal forms:

- If the coordinate is of the form "facet F of Σ plus vertex", its sign is determined by the fact that in any convex realization of Σ, all vertices not on F lie on the same side of F. Using an orientation of Σ, this allows us to find a permutation of σ whose sign is positive in any convex realization of Σ.
- If σ contains no facet of Σ, its sign could be positive or negative in different realizations, so we simply order it lexicographically. These coordinates are referred to as *undetermined*.

From now on, all coordinates will be normalized in this way.

Suppose that two Grassmann–Plücker relations Γ_1, Γ_2 share exactly one undetermined coordinate σ which occurs in terms of different sign, so that

$$\Gamma_1 = \cdots + \tau_1 \cdot \sigma + \ldots,$$
$$\Gamma_2 = \cdots - \tau_2 \cdot \sigma + \ldots$$

for some coordinates τ_1, τ_2. In this situation we can eliminate σ from Γ_1 and Γ_2 by forming the combination

$$\tau_2\Gamma_1 + \tau_1\Gamma_2.$$

Such an elimination corresponds to an edge in a graph whose nodes are all possible Grassmann–Plücker relations. If we succeed in finding a tree on this node set such that after elimination, all remaining terms are positive, we will have found our certificate of non-realizability: All terms in the result of the elimination are positive in any convex realization, but this elimination is a combination of Grassmann–Plücker relations that are zero in any realization.

Before sketching some of the details that make this implementation fast, we outline the basic steps.

1. Enumerate all Grassmann–Plücker relations with exactly one undetermined coordinate. Each of these relations is used as a leaf of a putative tree, and a collection $\mathcal{C}$ of candidate trees is initialized with these leaves. (In contrast to [16], we now use *all* Grassmann–Plücker relations (1), not just the three-term ones.)
2. For each leaf such that the term containing the undetermined coordinate σ is $\pm\tau \cdot \sigma$, enumerate *all* Grassmann–Plücker relations that contain a term of the form $\mp\tau' \cdot \sigma$ (not just those with exactly one undetermined coordinate), and add the resulting two-node trees to $\mathcal{C}$.
3. Iteratively, for each tree in $\mathcal{C}$ that contains a relation with an unpaired term $\pm\tau \cdot \sigma$, look for trees that contain an unpaired term of the form $\mp\tau' \cdot \sigma$, combine them, and store the result.
4. Continue until either a tree with no remaining unpaired undetermined terms is found, or all available memory is exhausted.

3.2 Optimizations

Several details are crucial to achieving good performance.

Efficient enumeration of Grassmann–Plücker relations It is absolutely crucial to avoid enumerating all possible Grassmann–Plücker relations. Our code first enumerates all coordinates σ that do not contain a facet, and then carefully only generates those relations Γ that contain a term with σ.

Encoding of data Each Grassmann–Plücker relation $\Gamma(I, J)$ is currently represented internally by a 64-bit integer whose top 32 bits contain I and whose lower 32 bits contain J.[2] If need be,

[2]In actual fact, we currently store $\pm\Gamma(I, J)$ as a signed 64 bit integer and only allow up to 31 vertices; so one bit goes unused. This will become irrelevant as soon as more general bitsets are used to store the index sets.

this limitation on the number of vertices can be overcome in the future by using larger bitset classes. However, this encoding permits the basic data structures like vectors, maps and hash sets to operate on *integers* only instead of on more complicated data structures, which helps a lot to speed up the code. Similiarly, each normal form of a Grassmann–Plücker coordinate is also represented by an integer that is interpreted as a bitset.

Caching Each Grassmann–Plücker relation is calculated only once, and the result is cached to avoid repeated calculation of the normal form of coordinates. The set of relations containing a fixed coordinate is also cached and organized in a lookup-friendly data structure.

4 Results and discussion

Example 2. In [4], Francisco Criado and Francisco Santos constructed a family of *topological prismatoids*, a combinatorial abstraction of the geometric prismatoids used by Santos [20] to construct counterexamples to the Hirsch conjecture. Each of their topological prismatoids implies the existence of a combinatorial sphere whose diameter exceeds the Hirsch bound. The family is constructed by randomly flipping away from a seed prismatoid, and the members end up having between 14 and 28 vertices. The non-realizability of the four 14-vertex members was certified by [16] and independently by [9].

# vertices	# proven nonrealizable	# not proven nonrealizable	min size	max size
14	4	0	2	3
15	31	5	2	9 (#2173)
16	132	77	2	17 (#2786)

Table 1: Nonrealizability certificates for small topological prismatoids

Table 1 shows some preliminary results for small topological prismatoids. We can observe that in all cases, some examples have a small certificate of size 2. This pattern repeats itself for larger prismatoids, all the way up to the maximum size: for some examples, a small certificate is encountered almost immediately, while for others, all available memory is exhausted.

Note that [9] finds certificates for *all* prismatoids on 15 vertices, yet their method cannot handle the larger instances. In contrast, our method can immediately find some certificates even for prismatoids on 28 vertices, yet misses even some on 15 vertices. ◇

This example highlights some of the differences between our new implementation and [9]:

Size of data Since [9] represents slack polynomials explicitly as sums of monomials, the dimension d and the number n of vertices directly impact their performance. On the one hand, they enjoy the freedom of being able to work with general polynomials of degree m in $N = nd$ variables (one variable for each coordinate of each vertex), but on the other hand each such polynomial has $\binom{m+nd}{nd}$ monomials, and this number grows quite quickly.

In contrast, our approach is restricted to Grassmann–Plücker polynomials, but each such polynomial is represented in $O(n)$ many bits, and the dimension only enters as an $O(d)$ upper bound on the number of terms in each polynomial (1). (Since Plücker coordinates vanish for repeated entries, the size of each Grassmann–Plücker relation depends on the size of $I \cap J$.)

Types of certificates found As just outlined, our method can process larger examples — yet Gouveia et al's method finds some certificates that ours misses, and some of their certificates are shorter than ours. We have yet to investigate the underlying reasons for this more deeply.

References

[1] K. A. Adiprasito and A. Padrol. "The universality theorem for neighborly polytopes". In: *Combinatorica* 37.2 (2017), pp. 129–136. DOI: 10.1007/s00493-016-3253-9.

[2] J. Bokowski and B. Sturmfels. *Computational synthetic geometry*. Vol. 1355. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1989, pp. vi+168. DOI: 10.1007/BFb0089253.

[3] M. Brandt and A. Wiebe. "The slack realization space of a matroid". In: *Algebr. Comb.* 2.4 (2019), pp. 663–681. DOI: 10.5802/alco.68.

[4] F. Criado and F. Santos. "Topological Prismatoids and Small Simplicial Spheres of Large Diameter". In: *Experimental Mathematics* (2019), pp. 1–13. DOI: 10.1080/10586458.2019.1641766. eprint: https://doi.org/10.1080/10586458.2019.1641766.

[5] E. Gawrilow and M. Joswig. "polymake: a framework for analyzing convex polytopes". In: *Polytopes—combinatorics and computation (Oberwolfach, 1997)*. Vol. 29. DMV Sem. Birkhäuser, Basel, 2000, pp. 43–73.

[6] J. E. Goodman and R. Pollack. "Upper bounds for configurations and polytopes in $\mathbb{R}^d$". In: *Discrete Comput. Geom.* 1.3 (1986), pp. 219–227. DOI: 10.1007/BF02187696.

[7] J. Gouveia, A. Macchia, R. R. Thomas, and A. Wiebe. "The slack realization space of a polytope". In: *SIAM J. Discrete Math.* 33.3 (2019), pp. 1637–1653. DOI: 10.1137/18M1233649.

[8] J. Gouveia, A. Macchia, and A. Wiebe. *Combining realization space models of polytopes*. 2020. DOI: 10.48550/ARXIV.2001.11999.

[9] J. Gouveia, A. Macchia, and A. Wiebe. "General non-realizability certificates for spheres with linear programming". In: *J. Symbolic Comput.* 114 (2023), pp. 172–192. DOI: 10.1016/j.jsc.2022.04.013.

[10] G. Kalai. "Many triangulated spheres". In: *Discrete Comput. Geom.* 3.1 (1988), pp. 1–14. DOI: 10.1007/BF02187893.

[11] A. Macchia and A. Wiebe. "Slack ideals in Macaulay2". In: *Mathematical software—ICMS 2020*. Vol. 12097. Lecture Notes in Comput. Sci. Springer, Cham, 2020, pp. 222–231. DOI: 10.1007/978-3-030-52200-1_22.

[12] D. Maclagan and B. Sturmfels. *Introduction to tropical geometry*. Vol. 161. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2015, pp. xii+363. DOI: 10.1090/gsm/161.

[13] E. Miller and B. Sturmfels. *Combinatorial commutative algebra*. Vol. 227. Graduate Texts in Mathematics. Springer-Verlag, New York, 2005, pp. xiv+417.

[14] N. E. Mnëv. "The universality theorems on the classification problem of configuration varieties and convex polytopes varieties". In: *Topology and geometry—Rohlin Seminar*. Vol. 1346. Lecture Notes in Math. Springer, Berlin, 1988, pp. 527–543. DOI: 10.1007/BFb0082792.

[15] E. Nevo, F. Santos, and S. Wilson. "Many triangulated odd-dimensional spheres". In: *Math. Ann.* 364.3-4 (2016), pp. 737–762. DOI: 10.1007/s00208-015-1232-x.

[16] J. Pfeifle. "Positive Plücker tree certificates for non-realizability". In: *Experimental Mathematics* 0.0 (2022), pp. 1–17. DOI: 10.1080/10586458.2021.1994487.

[17] J. Pfeifle and G. M. Ziegler. "Many triangulated 3-spheres". In: *Math. Ann.* 330.4 (2004), pp. 829–837. DOI: 10.1007/s00208-004-0594-2.

[18] J. Richter-Gebert. "Two interesting oriented matroids". In: *Doc. Math.* 1 (1996), No. 07, 137–148.

[19] J. Richter-Gebert and G. M. Ziegler. "Realization spaces of 4-polytopes are universal". In: *Bull. Amer. Math. Soc. (N.S.)* 32.4 (1995), pp. 403–412. DOI: 10.1090/S0273-0979-1995-00604-X.

[20] F. Santos. "A counterexample to the Hirsch conjecture". In: *Ann. of Math. (2)* 176.1 (2012), pp. 383–412. DOI: 10.4007/annals.2012.176.1.7.

[21] P. W. Shor. “Stretchability of pseudolines is NP-hard”. In: *Applied geometry and discrete mathematics*. Vol. 4. DIMACS Ser. Discrete Math. Theoret. Comput. Sci. Amer. Math. Soc., Providence, RI, 1991, pp. 531–554.

[22] H. Zheng. “Ear decomposition and balanced neighborly simplicial manifolds”. In: *Electron. J. Combin.* 27.1 (2020), Paper No. 1.10, 17.

Generalized Perron roots and solvability of the absolute value equation

Manuel Radons[1*], Josué Tonelli-Cueto[2†]

[*]Technische Universität Berlin, Chair of Discrete Mathematics/Geometry
[†]IMJ-PRG, INRIA Paris

Abstract

Let A be a real $(n \times n)$-matrix. The piecewise linear equation system $z - A|z| = b$ is called an absolute value equation (AVE). It is well-known to be equivalent to the linear complementarity problem (LCP). For AVE and LCP unique solvability is comprehensively characterized in terms of conditions on the spectrum (AVE), resp., the principal minors (LCP) of the coefficient matrix. For mere solvability no such characterization exists. We close this gap in the theory on the AVE-side. The aligning spectrum of A consists of real eigenvalues of the matrices SA, where $S \in \mathrm{diag}(\{\pm\}^n)$, which have a corresponding eigenvector in the positive orthant of $\mathbb{R}^n$. For the mapping degree of the piecewise linear function $z \mapsto z - A|z|$ we prove, under some mild genericity assumptions on A: The degree is 1 if all aligning values are smaller than 1, it is 0 if all aligning values are larger than 1, and in general it is congruent to $(k+1) \mod 2$ if k aligning values are larger than 1. The modulus cannot be omitted because the degree can both increase and decrease.

1 Introduction

The linear complementarity problem (LCP) stands at the crossroads of numerous optimization contexts. Not only do many problems in computational mechanics arise naturally in LCP form. Linear and quadratic programs are special cases of the LCP. For a comprehensive introduction to the topic, see [7]. Other relevant optimization problems, such as bimatrix games [7], or arbitrary finite piecewise affine systems [8], can be reduced to solving an LCP.

Linear complementarity problems can be formulated in various equivalent ways, e.g., via complementarity conditions, max-min expressions, or absolute values. In terms of the wealth of publications treating them, and the depth of the associated theory, two formulations stand out. First, the classical form by Cottle and Dantzig, which was first introduced in [6]: Let $M \in \mathrm{M}_n(\mathbb{R})$, where $\mathrm{M}_n(\mathbb{R})$ denotes the space of $n \times n$ real matrices, and $q \in \mathbb{R}^n$. Then the $\mathrm{LCP}(M, q)$ is to find vectors $v, w \in \mathbb{R}^n_{\geq 0}$ with $w^T v = 0$ so that

$$w \;=\; Mv + q\,.$$

Now let $A \in \mathrm{M}_n(\mathbb{R})$ and $b \in \mathbb{R}^n$. Then the piecewise linear equation system

$$z - A|z| \;=\; b\,,$$

where $|\cdot|$ denotes the componentwise absolute value, is the second outstanding formulation, called an absolute value equation $\mathrm{AVE}(A, b)$. The term was first coined by Mangasarian in [9], but the first journal publication to investigate it was [13].

[1]Email: radons@math.tu-berlin.de.
[2]josue.tonelli-cueto@inria.fr.

Setting $x := v - w$, and consequently $v = \max(0, x)$, and $w = -\min(0, x)$, both formulations can be transformed into another via the identities $\max(0, x) = (|x| + x)/2$ and $\min(0, x) = (x - |x|)/2$ [10]. For both AVE and LCP unique solvability is comprehensively characterized.

The LCP(M, q) is uniquely solvable for arbitrary $q \in \mathbb{R}^n$ if and only if M is a P-matrix, that is, if all its principal minors are positive [7]. Let $\mathcal{S}$ be the set of $n \times n$ signature matrices, i.e., diagonal matrices with entries in $\{-1, +1\}$. Then the largest real eigenvalue of any matrix SA, where $S \in \mathcal{S}$, is called the sign-real spectral radius of A. The AVE(A, b) is uniquely solvable if and only if the sign-real spectral radius of A is smaller than one [10, 14].

The sign-real spectral radius can be interpreted as a piecewise linear analogue of contractivity conditions for linear operators, or as a generalized Perron root for matrices without sign-restrictions [14]. Its computation is equivalent to the computation of the weighted componentwise distance to the nearest singular matrix [14]. The latter relation provides a direct connection between the condition of the matrices $I - SA$ and the complexity of the AVE, which is NP-complete in general [5], but lies in $\mathrm{O}(n^3)$ for certain uniquely solvable systems with well-conditioned matrices $I - SA$ [12]. The latter fact makes the AVE particularly interesting in light of recent developments in real algebraic geometry that deal with precisely such connections of complexity and condition, see [2, 3] for founding texts of the research area, and [4, 1] for recent applications.

For mere, possibly non-unique, solvability of AVE and LCP there exists no similarly comprehensive characterization as in the case of unique solvability. We close this gap in the theory on the AVE side. We again consider nonnegative real eigenvalues of the matrices SA, but with the restriction that only those are taken into account that have a corresponding eigenvector in the positive orthant of $\mathbb{R}^n$. The resulting eigenvalues are called aligning values. The set of aligning values, ordered by descending magnitude, is called the aligning spectrum. We denote the largest and the smallest aligning value $\lambda_{\max}$ and $\lambda_{\min}$, respectively. We prove that, under some genericity assumptions on A, the mapping degree of the piecewise linear function

$$F_A : \mathbb{R}^n \to \mathbb{R}^n, \quad z \mapsto z - A|z|$$

is one if $\lambda_{\max} < 1$, it is zero if $\lambda_{\min} > 1$, and otherwise we have

$$\deg F_A \equiv (k + 1) \mod 2,$$

where k is the number of aligning values larger than one. The genericity of A can always be achieved by a perturbation of A, which does not change the degree of F_A, as the degree is locally constant in the function space.

Content and structure: In the subsequent section we introduce some basics to our needs.

2 Preliminaries - degree basics

In this section we present a collection of basic facts about the mapping degree that is tailored to our needs. All statements are taken directly from or are special cases of theorems in [11, p. 111 ff].

Let U be some nonempty connected subset of $\mathbb{R}^n$. We say a function on f is almost smooth if it is continuous and differentiable almost everywhere. Let $f : U \to U$ be a function which is almost smooth, and for $y \in U$ let $f^{-1}(y) = \{x \in U : f(x) = y\}$ be the fiber of y under f. We say y is a regular value of f if f is differentiable with nonsingular Jacobian $d_x f$ at all $x \in f^{-1}(y)$. This includes the case that $f^{-1}(y) = \emptyset$.

If f is proper, that is, if preimages of compact sets under f are compact, then the oriented preimage count

$$\sum_{x \in f^{-1}(y)} \operatorname{sign}(\det(d_x f))$$

of any regular value y of f equals a constant d, which is called the (mapping) degree of f. By the Theorem of Sard Brown regular values lie dense in the range of an almost smooth function. Hence, for closed U and closed f, a nonzero degree implies surjectivity of f.

Note that properness is a sufficient, not a necessary condition for a consistent degree. Any constant function on $\mathbb{R}^n$ has degree 0 without being proper. A positively homogeneous function $F : \mathbb{R}^n \to \mathbb{R}^n$ is proper if and only if it is nondegenerate in the sense that it maps no nonzero point to the origin. If one is only interested in degree-related questions, it makes sense to force a nondegenerate positively homogeneous function onto the Euclidean unit sphere $\mathbb{S}^{n-1}$ via

$$\bar{F} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}, \quad x \mapsto \frac{F(x)}{\|F(x)\|_2}. \tag{1}$$

An almost smooth function on $\mathbb{S}^{n-1}$ is proper by default due to the compactness of the sphere and thus has a well defined mapping degree. One can show that the degree of F and its normalization $\bar{F}$ coincide, cf. [7, Chap. 6.2].

In the context of this work, a proper homotopy is an almost smooth function $H : U \times [0,1] \to U$ so that for all $t \in [0,1]$ the function $H_t \coloneqq H(\cdot, t)$ is proper. Two proper almost smooth functions F, G on U are properly homotopic if there exists a proper homotopy H so that $H_0 = F$ and $H_1 = G$. The mapping degree is invariant under proper homotopies. We close this subsection with an important fact.

Lemma 1. *Let $F, G : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be continuous mappings so that*

$$\|F(x) - G(x)\|_2 < 2$$

for all $x \in \mathbb{S}^{n-1}$. Then the mapping degrees of F and G coincide.

This is due to the fact that the homotopy

$$H : \mathbb{S}^{n-1} \times [0,1] \to \mathbb{S}^{n-1}, (x,t) \mapsto \frac{tF(x) + (1-t)G(x)}{\|tF(x) + (1-t)G(x)\|_2}$$

is then well defined, as the term in the denominator on the right-hand side cannot become 0.

3 The aligning spectrum

The piecewise linear function

$$F_A : \mathbb{R}^n \to \mathbb{R}^n, \quad z \mapsto z - A|z|$$

is linear and thus differentiable on the orthants of $\mathbb{R}^n$. The locus of non-differentiability are the orthant boundaries. It is thus almost smooth and positively homogeneous. For our investigation, we need to answer when F_A is nondegenerate and thus proper, so that we can define the normalization $\bar{F}_A$.

Lemma 2. *Let $A \in \mathrm{M}_n(\mathbb{R})$. Then F_A is degenerate if and only if there exists a $S \in \mathcal{S}$ so that SA has a real eigenvalue 1 with corresponding nonnegative eigenvector.*

Proof. "⇒": Let $z \in \mathbb{R}^n$ be a solution of $z - A|z| = 0$ and let S be a signature of z, so that $Sz = |z|$. Then we have

$$0 = Sz - SA|z| = |z| - SA|z| \iff |z| = SA|z|.$$

"⇐": Let v be the eigenvector of SA in question. Then we have $v = |v|$ and thus

$$0 = v - SAv = v - SA|v| \iff 0 = Sv - A|v| = Sv - A|Sv|,$$

which concludes the proof. □

This lemma provides some crucial insights into $\bar{F}_A$, which will be the key motivation for our definition of the aligning spectrum:

Corollary 3. *Let λ be a positive real eigenvalue of some SA which has a corresponding nonnegative eigenvector v with $\|v\|_2 = 1$. Then, if F_A is nondegenerate,*

1. *$-v$ is a fixed point of $\bar{F}_A$,*
2. *v is a fixed point of $\bar{F}_A$, if $\lambda < 1$,*
3. *v is mapped to $-v$ by $\bar{F}_A$, if $\lambda > 1$,*

and $\bar{F}_A$ has no fixed points or points that are mapped to their antipode which are not characterized in this fashion.

So the interesting objects of investigation for the analysis $F_A, \bar{F}_A$, and homotopies of the latter induced by scalings of A are vectors $z \in \mathbb{R}^n$ so that z and $A|z|$ are aligned. Lemma 2 and Corollary 3 thus justify:

Definition 4. Let $A \in \mathrm{M}_n(\mathbb{R})$ and $S \in \mathcal{S}$. We call a nonnegative real eigenvalue λ of SA an aligning value of A if there exists a nonnegative eigenvector corresponding to λ. We call such an eigenvector an aligning vector of A. The set of aligning values, enumerated by descending magnitude, is the aligning spectrum of A, denoted

$$\mathrm{Spec}^a(A) \;=\; \{\lambda_1, \ldots, \lambda_\ell\}\,.$$

We further set $\lambda_{\max}(A) \coloneqq \lambda_1$ and $\lambda_{\min}(A) \coloneqq \lambda_\ell$.

We note that the aligning spectrum cannot be empty since a continuous mapping on the sphere has at least one fixed point. Set

$$\bar{H}_A :\; \mathbb{S}^{n-1} \times [0,1] \;\to \mathbb{S}^{n-1}\,, \quad z \;\mapsto\; \bar{F}_{tA}\,. \tag{2}$$

Lemma 2 yields:

Lemma 5. *Let $A \in \mathrm{M}_n(\mathbb{R})$. Then the homotopy $\bar{H}_A$ from (2) is proper if and only if the interval $[0,1]$ does not contain a reciprocal of an aligning value of A.*

This leads to our first main result.

Theorem 6. *Let $A \in \mathrm{M}_n(\mathbb{R})$. Then the mapping degree of F_A equals 1 if $\lambda_{\max}(A) < 1$. Further, if A is nonsingular and $\lambda_{\min}(A) > 1$, then the mapping degree of F_A is 0.*

Proof. The first part of the statement is an immediate consequence of Lemma 5 and the correspondence of the degree of F_A and $\bar{F}_A$. Alternatively, consider that $\bar{F}_A$ maps no point to its polar opposite and apply Lemma 1.

Concerning the second part: Let A be nonsingular. Then there exist a hyperplane V and an open halfspace V^+ so that all column vectors of A are contained in V^+. Since $\lambda_{\min} > 1$, F_A and F_{tA} are properly homotopic for all $t \geq 1$. Let $S \in \mathcal{S}$. Then there exists a $t_S > 1$ so that the columns of the matrix $S - tA$ lie in V^+ for all $t \geq t_S$. Hence, for any t larger than the maximum over the t_S, the image of all orthants of $\mathbb{R}^n$ under F_{tA} must be contained in $V^+ \cup \{0\}$. But then F_A is properly homotopic to a function which cannot be surjective and must thus have degree 0. □

Corollary 7. *Let $A \in \mathrm{M}_n(\mathbb{R})$ so that $\lambda_{\max}(A) = 0$. Then for all $t \in \mathbb{R}$ the function F_A has degree 1.*

Proof. For any $t \in \mathbb{R}$ the function F_A is properly homotopic to F_{tA}. □

4 Summary of the rest of the work

The key observation underlying this section is that there are stable and unstable aligning values. And only the stable ones are relevant to the determination of the mapping degree of F_A. Let

$$D_\varepsilon := \begin{pmatrix} 1 & -0.5-\varepsilon \\ 0.5 & 0 \end{pmatrix}. \tag{3}$$

We have $\lambda_{\max}(D_0) = 0.5$, but for any $\varepsilon > 0$ the corresponding eigenvalue becomes complex and we get $\lambda_{\max}(D_\varepsilon) = 0.5(\sqrt{2}\sqrt{1+\varepsilon} - 1)$, which roughly equals 0.207 for small ε.

The degree of F_{tD_0} equals 1 for $t = 1$ and $t = 2.1$. That is, it does not flip when $\lambda_{\max}(tD_0)$ becomes larger than 1. But for t so that $t(\sqrt{2}\sqrt{1+\varepsilon} - 1) > 1$, the degree of F_{tD_0} becomes 0. In light of the fact that the mapping degree is stable under perturbations, this obsevation makes perfect sense, and it leads us to the following:

Definition 8. Let $A \in \mathrm{M}_n(\mathbb{R})$. We say A is generic if all its aligning values are simple and the corresponding aligning vectors do not lie in an orthant boundary.

We prove, in essence, that a generic form can be achieved with probability 1 by a random perturbation (which does not affect the degree), as non-generic matrices are confined to some lower-dimensional surface in $\mathrm{M}_n(\mathbb{R})$. This leads to the main result.

Theorem 9. *Let $A \in \mathrm{M}_n(\mathbb{R})$ with $1 \notin \mathrm{Spec}^a(A)$ be generic, and let k be the number of aligning values of A which are larger than 1. Then we have*

$$\deg F_A \;\equiv\; (k+1) \mod 2\,.$$

The proof is fairly involved and cannot be presented here. For $k = 0$ the statement is correct due to Theorem 6. The basic idea for an induction is to compare two functions $\bar{F}_{t_1 A}$, $\bar{F}_{t_2 A}$ on the sphere, where t_1 and t_2 are in a small neighborhood of the reciprocal of an aligning value, t_1 being smaller, t_2 larger than the reciprocal. Then Lemma 1 is used to construct a function $G : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ which is homotopic to $\bar{F}_{t_2 A}$, but much simpler to analyze. It is then shown that the mapping degree of G differs from that of $\bar{F}_{t_1 A}$ by exactly 1, which establishes the theorem.

It is both possible that the degree increases and decreases when moving from $\bar{F}_{t_1 A}$ to $\bar{F}_{t_2 A}$. Hence, the modulus operator cannot be omitted from the statement of Theorem 9.

References

[1] "Algebra meets numerics: Condition and complexity". In: `https://www.math.tu-berlin.de/fachgebiete_ag_diskalg/fachgebiet_algorithmische_algebra/v_menue/veranstaltungen/2017ws/algebra_meets_numerics_condition_and_complexity`.

[2] D. Amelunxen and M. Lotz. "Average-case complexity without the black swans". In: *J. Complexity* 41 (2017), pp. 82–101. DOI: `10.1016/j.jco.2016.12.002`.

[3] P. Bürgisser and F. Cucker. *Condition - The geometry of numerical algorithms*. Vol. 349. Springer, Heidelberg, 2013, pp. xxxii+554. DOI: `10.1007/978-3-642-38896-5`.

[4] P. Bürgisser, F. Cucker, and J. Tonelli-Cueto. "Computing the homology of semialgebraic sets. I: Lax formulas". In: *Found. Comput. Math.* 20.1 (2020), pp. 71–118. DOI: `10.1007/s10208-019-09418-y`.

[5] S. J. Chung. "NP-completeness of the linear complementarity problem". In: *J. Optim. Theory Appl.* 60.3 (1989), pp. 393–399. DOI: `10.1007/BF00940344`.

[6] R. W. Cottle and G. B. Dantzig. “Complementary pivot theory of mathematical programming”. In: *Linear Algebra Appl.* 1.1 (1968), pp. 103–125. DOI: 10.1016/0024-3795(68)90052-9.

[7] R. W. Cottle, J.-S. Pang, and R. E. Stone. *The linear complementarity problem*. Computer Science and Scientific Computing. Academic Press, Inc., Boston, MA, 1992, pp. xxiv+762.

[8] A. Griewank, J.-U. Bernt, M. Radons, and T. Streubel. “Solving piecewise linear systems in abs-normal form”. In: *Linear Algebra Appl.* 471 (2015), pp. 500–530. DOI: 10.1016/j.laa.2014.12.017.

[9] O. L. Mangasarian. “Absolute value programming”. In: *Comput. Optim. Appl.* 36.1 (2007), pp. 43–53. DOI: 10.1007/s10589-006-0395-5.

[10] A. Neumaier. *Interval methods for systems of equations*. Vol. 37. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1990, pp. xvi+255.

[11] E. Outerelo and J. M. Ruiz. *Mapping degree theory*. Vol. 108. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2009, pp. x+244. DOI: 10.1090/gsm/108.

[12] M. Radons. “Direct solution of piecewise linear systems”. In: *Theoret. Comput. Sci.* 626 (2016), pp. 97–109. DOI: 10.1016/j.tcs.2016.02.009.

[13] J. Rohn. “Systems of linear interval equations”. In: *Linear Algebra Appl.* 126 (1989), pp. 39–78. DOI: 10.1016/0024-3795(89)90004-9.

[14] S. M. Rump. “Theorems of Perron-Frobenius type for matrices without sign restrictions”. In: *Linear Algebra Appl.* 266 (1997), pp. 1–42. DOI: 10.1016/S0024-3795(96)00522-8.

Normal limiting distributions for systems of linear equations in random sets

Juanjo Rué[1*], Maximilian Wötzel[2†]

[*]Departament de Matemàtiques and Institut de Matemàtiques de la UPC (IMTech), Universitat Politècnica de Catalunya & Centre de Recerca Matemàtica & Barcelona Graduate School of Mathematics (BGSMath), Barcelona, Spain.
[†]Department of Mathematics, Radboud University, Nijmegen, The Netherlands.

The full version of this work can be found in [12].

Abstract

We consider the binomial random set model $[n]_p$ where each element in $\{1,\dots,n\}$ is chosen independently with probability $p := p(n)$. We show that for essentially all regimes of p and very general conditions for a matrix A and a column vector $\mathbf{b}$, the count of specific integer solutions to the system of linear equations $A\mathbf{x} = \mathbf{b}$ with the entries of $\mathbf{x}$ in $[n]_p$ follows a (conveniently rescaled) normal limiting distribution. This applies among others to the number of solutions with every variable having a different value, as well as to a broader class of so-called non-trivial solutions in homogeneous strictly balanced systems. Our proof relies on the delicate linear algebraic study both of the subjacent matrices and the corresponding ranks of certain submatrices, together with the application of the method of moments in probability theory.

The study of the existence of solutions to linear equations in subsets of the integers (and more generally in additive or even non-abelian groups) has been a prominent topic not only in number theory, but also in extremal combinatorics, ergodic theory, functional analysis and theoretical computers science, among other research areas. A prototypical example of such investigations is Roth's Theorem [8], which proves (by using Fourier analytic means) that dense sets of integers always contains arithmetic progressions of length 3. This result was fully generalized by Szemerédi [18], who obtained a similar statement for arithmetic progressions of all (fixed) lengths.

Following this line of research, Frankl, Graham and Rödl [5, Theorem 2] proved similar results for homogeneous systems of linear equations. In particular, they proved that when A is a *density regular* matrix with integer entries (that is, the columns of A sum up to the zero vector), then any dense integer set T will contain solutions to the linear system $A\mathbf{x} = 0$. This result generalizes Szemerédi's Theorem, as arithmetic progressions of length k (or k-APs for short) can be encoded as solutions to the linear system $x_2 - x_1 = x_3 - x_2 = \cdots = x_k - x_{k-1}$. Other classical equations, such as the Schur equation for sum-free sets (sets without solutions to the equation $x + y = z$), and Sidon sets (sets without non-trivial solutions to the equation $x + y = z + t$) also fit into this framework.

[1]Email: juan.jose.rue@upc.edu. Research of J. R. supported by the Spanish State Research Agency through projects MTM2017-82166-P, PID2020-113082GB-I00, the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M), and the Marie Curie RISE research network 'RandNet' MSCA-RISE-2020-101007705.

[2]Email: m.woetzel@math.ru.nl. This research was conducted while M. W. was a member of the Barcelona Graduate School of Mathematics (BGSMath) as well as the Universitat Politècnica de Catalunya. Maximilian Wötzel acknowledges financial support from the Fondo Social Europeo and the Agencia Estatal de Investigación through the FPI grant number MDM-2014-0445-16-2 and the Spanish Ministry of Economy and Competitiveness, through the María de Maeztu Programme for Units of Excellence in R&D (MDM-2014-0445), through the project MTM2017-82166-P, as well as from the Dutch Science Council (NWO) through the grant number OCENW.M20.009.

A very recent trend of investigation is to transfer these extremal results to to a sparse setting, and hence to prove analogues on subsets which are dense relative to these ambient sets. Specific examples for such sparse ambient sets are for instance the primes or perfect powers, but also the binomial random set $[n]_p$ obtained by choosing to include independently each element in $[n] = \{1, 2, \dots, n\}$ with probability p. Returning to the example of k-APs, define a set T to be (δ, k)–*Szemerédi* if every subset $U \subseteq T$ with at least $\delta|T|$ elements contains a k-AP. Thus, Szemerédi's Theorem shows that $[n]$ (in fact, every set T that is itself dense in $[n]$) is (δ, k)–Szemerédi for each δ and k. For the specific case of $k = 3$, Kohayakawa, Łuczak and Rödl [7] proved that for all $\delta > 0$ there exists constants $c, C > 0$ such that if $p \leq cn^{-1/2}$, then as n tends to infinity, $\Pr([n]_p$ is $(\delta, 3)$–Szemerédi) tends to 0, while if $p \geq Cn^{-1/2}$, it tends to 1. The existence of this threshold was extended to all values of k in [15, 4], and later rediscovered in [14, 1] in the context of independent sets in hypergraphs. These ideas, named *hypergraph container method* were then used by Spiegel [16] and independently by Hancock, Staden and Treglown [6] to extend [5, Theorem 2] to the broadest class of linear systems possible. Similar techniques were used by Rué, Serra and Vena [10] to study random sparse analogues of [5, Theorem 2] in finite fields and more general configurations than linear systems of equations.

The main contribution of our paper follows this trend of research, with the aim of studying the *typical* number of solutions to linear systems of equations in random sets of integers. The most general version of our results is very technical and stating it would require the introduction of a wide variety of algebraic definitions and lemmas, so we omit it in this extended abstract and instead refer to the full version of this work [12]. However, we will now introduce some main definitions in order to be able to formulate two important consequences, namely Theorems 4 and 7.

Proper solutions and their distribution. In order to properly state our results we need to introduce some notation. Let $m > r$ be positive integers, $A \in \mathbb{Z}^{r\times m}$ an integer matrix, and $\mathbf{b} \in \mathbb{Z}^r$ an integer vector. We write the *rank* of a matrix A by $\mathrm{rk}(A)$. Define $S(A, \mathbf{b}) = \{\mathbf{x} \in \mathbb{Z}^m : A\mathbf{x} = \mathbf{b}\}$ as the set of integer solutions to the system of linear equations $A\mathbf{x} = \mathbf{b}$. Furthermore, if $Q \subset [m]$, let $A^Q \in \mathbb{Z}^{r\times|Q|}$ denote the $r \times |Q|$ matrix obtained by only keeping the columns of A indexed by Q. We notice that the rank of the empty matrix $A^\emptyset$ is zero. We will identify tuples $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m$ with the corresponding column vectors but abuse notation slightly by letting $\mathbf{x}^Q$ denote the vector obtained by only keeping the *rows* of $\mathbf{x}$. This should not be confused with the notation $\mathbf{x}^k$, where k is a positive integer, which we will use to denote the vector

$$\mathbf{x}^k = (\underbrace{\mathbf{x}, \dots, \mathbf{x}}_{k \text{ times}}) = (x_1, \dots, x_m, x_1, \dots, x_m, \dots, x_1, \dots, x_m) \in \mathbb{Z}^{km}.$$

Finally, if $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m$ is an m-tuple, we will write $\{\mathbf{x}\}$ as a shorthand for the set $\{x_1, \dots, x_m\}$. In particular, if some entry in $\mathbf{x}$ is repeated, then the cardinality of $\{\mathbf{x}\}$ is strictly less than m.

Roughly speaking, our main goal in this paper is to estimate $|S(A, \mathbf{b}) \cap [n]_p^m|$, that is, the number of solutions whose entries lie in $[n]_p$ for different regimes of p. In order for this to be a well-posed problem, we need some definitions from [17] and [11].

Definition 1. An integer matrix $A \in \mathbb{Z}^{r\times m}$ is said to be

i) *positive* if there exists an integer solution to $A\mathbf{x} = \mathbf{0}$ having all positive entries:

$$S(A, \mathbf{0}) \cap \mathbb{N}^m \neq \emptyset,$$

and if for any pair of indices $i, j \in [m]$, $i \neq j$, there exists a solution $(x_1, \dots, x_m) \in S(A, \mathbf{0})$ satisfying $x_i \neq x_j$.

ii) *abundant* if $\mathrm{rk}(A) > 0$ and the removal of at most two columns does not change the rank of A:

$$\mathrm{rk}(A^Q) = \mathrm{rk}(A)$$

for any $Q \subset [m]$ satisfying $|Q| \geq m - 2$.

Sometimes the second requirement for positivity in Definition 1 is called *irredundancy*. For our applications, we will never consider positivity and irredundancy separately, and hence we have combined those two properties into a single one for expediency's sake.

In general, considering any possible solution in $S(A, \mathbf{b})$ is a bit too lenient. For instance, in the 3-AP situation (whose associated matrix is $A = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}$), $S(A, \mathbf{0})$ also contains all the tuples (a, a, a) with $a \in \mathbb{Z}$, and those are clearly not of interest. To remedy this issue we need to introduce the easy notion of *proper* solutions to $A\mathbf{x} = \mathbf{b}$.

Definition 2. Let $m > r$ be positive integers, $A \in \mathbb{Z}^{r\times m}$, and $\mathbf{b} \in \mathbb{Z}^r$. Then the set $S_0(A, \mathbf{b})$ of *proper* solutions is the subset of $S(A, \mathbf{b})$ with all coordinates being pairwise distinct, that is

$$S_0(A, \mathbf{b}) = \{(x_1, \dots, x_m) \in S(A, \mathbf{b}) : x_i \neq x_j \text{ for all } 1 \leq i < j \leq m\}.$$

Before proceeding to the statement of our first main result, we need to define one more parameter, which will measure the densest subsystem of A. The motivation behind this can be compared to the graph setting, where one wants to study the number of occurrences of some fixed graph G as a subgraph of a binomial random graph with n vertices and edge probability p. Here, one naively would expect the threshold for a random graph to contain G to be when p is around $n^{-v(G)/e(G)}$, which is when the expected number of copies of G flips from 0 to positive. But this is not the case: if G contains a subgraph H with $v(H)/e(H) > v(G)/e(G)$, then $n^{-v(H)/e(H)}$ will define the threshold instead. A similar behavior occurs for the distribution of subgraph counts, as shown by Ruciński in [9].

Definition 3. For positive integers $m > r$ and a positive integer matrix $A \in \mathbb{Z}^{r\times m}$, define the *density* $c(A)$ of A by

$$c(A) = \max_{\emptyset \neq Q \subset [m]} \frac{|Q|}{|Q| - r_Q},$$

where $r_Q = r_Q(A) = \mathrm{rk}(A) - \mathrm{rk}(A^{\overline{Q}})$, and $\overline{Q} = [m] \setminus Q$.

Note that this is indeed well-defined, since for a positive matrix it is not difficult to prove that $\mathrm{rk}(A^{\overline{Q}}) \geq \mathrm{rk}(A) - |Q| + 1$ for every $\emptyset \neq Q \subset m$.

We are now ready to state our first main theorem. For a random variable X with finite first moment $\mathbb{E}(X)$ and non-zero finite variance $\mathbb{V}\mathrm{ar}(X)$, denote by $\tilde{X} = (X - \mathbb{E}(X))/\sqrt{\mathbb{V}\mathrm{ar}(X)}$ its normalization. We write $X_n \xrightarrow{d} Y$ when a sequence of random variables $\{X_n\}_{n\geq 1}$ tends in distribution to Y.

Theorem 4. *Let $m > r$ be positive integers, $A \in \mathbb{Z}^{r\times m}$ a positive and abundant integer matrix, and $\mathbf{b} \in \mathbb{Z}^r$ such that $S(A, \mathbf{b}) \neq \emptyset$. Furthermore, let n be an integer, $0 \leq p := p(n) \leq 1$ and X_n the random variable equal $|S_0(A, \mathbf{b}) \cap [n]_p^m|$, which counts the number of proper solutions $\mathbf{x} \in [n]_p^m$ to $A\mathbf{x} = \mathbf{b}$. Then*

$$\tilde{X}_n \xrightarrow{d} \mathcal{N}(0, 1)$$

if $n(1-p) \to \infty$ and $np^{c(A)} \to \infty$.

Note that in the specific case of k-APs, this problem was already investigated by Barhoumi-Andréani, Koch and Liu [2]. In fact, they prove not only results on the limiting distribution of the number of k-term arithmetic progressions in $[n]_p$ even when k is unbounded (but sub-logarithmic), they also establish a bivariate central limit theorem for the joint distribution when considering the counts of two distinct progression lengths. In contrast, Theorem 4 in the setting of progressions requires the length to be fixed.

Non-trivial solutions, and main theorem for their distribution. Proper solutions are always of interest in the number theoretical context, but in a wide variety of situations we need to take care of non-proper solutions. For example, for $A = \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix}$, we see that $S(A, \mathbf{0})$ is the number of additive quadruples satisfying $a + b = c + d$. Clearly, if $\{a, b\} \cap \{c, d\} \neq \emptyset$, then these two sets must in fact be the same. On the other hand, solutions of the form $2a = c + d$ with $a \notin \{c, d\}$ are not proper but obviously of interest and might be considered as valid ones.

Keeping this in mind, we will now recall the notion of *non-trivial* solutions for systems of linear equations due to Rué, Spiegel and Zumalacárregui [11] which generalized an earlier notion of this for single line equations introduced by Ruzsa [13]. For a solution $\mathbf{x} = (x_1, \dots, x_m) \in S(A, \mathbf{b})$, we define $\mathfrak{p}(\mathbf{x}) \subset 2^{[m]}$ to be the partition of $[m]$ such that for any $i, j \in [m]$ it holds that $x_i = x_j$ if and only if i and j are in the same partition class of $\mathfrak{p}(\mathbf{x})$. One can view $\mathfrak{p}(\mathbf{x})$ as an ordered $|\mathfrak{p}(\mathbf{x})|$-tuple $(C_1, \dots, C_{|\mathfrak{p}(\mathbf{x})|})$ such that $\min C_i < \min C_j$ whenever $i < j$. Doing this, we can now define the matrix $A_{\mathfrak{p}(\mathbf{x})}$ in the following way. Suppose the columns of A are denoted by $\mathbf{c}_1, \dots, \mathbf{c}_m \in \mathbb{Z}^r$, then

$$A_{\mathfrak{p}(\mathbf{x})} = \Big(\textstyle\sum_{i \in C_1} \mathbf{c}_i \quad | \quad \sum_{i \in C_2} \mathbf{c}_i \quad | \quad \cdots \quad | \quad \sum_{i \in C_{|\mathfrak{p}(\mathbf{x})|}} \mathbf{c}_i \Big),$$

so $A_{\mathfrak{p}(\mathbf{x})} \in \mathbb{Z}^{r \times |\mathfrak{p}(\mathbf{x})|}$ is the $r \times |\mathfrak{p}(\mathbf{x})|$ matrix obtained by contracting all columns in the same partition class, with columns ordered by the minimum index in each class. Finally, define the set

$$\mathfrak{P}(A) = \{\mathfrak{p} \subset 2^{[m]} : \mathfrak{p} \text{ is a partition of } [m] \text{ and } \operatorname{rk}(A_\mathfrak{p}) = \operatorname{rk}(A)\}.$$

We are finally ready to introduce the notion of a *non-trivial* solution.

Definition 5. Let $m > r$ be positive integers, $A \in \mathbb{Z}^{r \times m}$, and $\mathbf{b} \in \mathbb{Z}^r$. Then the set $S_1(A, \mathbf{b})$ of *non-trivial* solutions is the subset of $S(A, \mathbf{b})$ with associated partitions coming from $\mathfrak{P}(A)$, that is

$$S_1(A, \mathbf{b}) = \{\mathbf{x} \in S(A, \mathbf{b}) : \mathfrak{p}(\mathbf{x}) \in \mathfrak{P}(A)\}.$$

This definition might seem quite arbitrary, but the interested reader is invited to read the discussions in [11] and [17] which show that, in some sense, it is quite natural in that it encompasses the natural notions of non-triviality for specific systems of linear equations studied in the literature. When we want to investigate the distribution of nontrivial solutions, we actually need to look at $c(A_\mathfrak{p})$ for any partition type $\mathfrak{p}$ that is to be considered. A special case in which it suffices to only consider $c(A)$ is that of *strictly balanced* systems.

Definition 6. Let $m > r$ be positive integers and $A \in \mathbb{Z}^{r \times m}$ be positive. Then the matrix A is called *strictly balanced* if

$$c(A) = \frac{m}{m - \operatorname{rk}(A)} > \max_{\emptyset \subsetneq Q \subsetneq [m]} \frac{|Q|}{|Q| - r_Q(A)},$$

and furthermore for every $\mathfrak{p} \in \mathfrak{P}(A) \setminus \{\{1\}, \{2\}, \dots, \{m\}\}$ such that $A_\mathfrak{p}$ is positive, it holds that $c(A) > c(A_\mathfrak{p})$.

We can prove the following theorem about the distribution of nontrivial solutions of strictly balanced systems of linear equations.

Theorem 7. *Let $m > r$ be positive integers, $A \in \mathbb{Z}^{r \times m}$ a positive and abundant strictly balanced integer matrix such that $S(A, \mathbf{0}) \neq \emptyset$. Furthermore, let n be an integer, $0 \leq p := p(n) \leq 1$ and X_n the random variable $|S_1(A, \mathbf{0}) \cap [n]_p^m|$ that counts the number of non-trivial solutions $\mathbf{x} \in [n]_p^m$ to $A\mathbf{x} = \mathbf{0}$. Then*

$$\tilde{X}_n \xrightarrow{d} \mathcal{N}(0, 1)$$

if $n(1 - p) \to \infty$ and $np^{c(A)} \to \infty$.

Both of our main results can be compared to those proved in [11] and [17]. In the former the authors investigated the threshold behavior of $|S_1(A, \mathbf{0}) \cap [n]_p^m|$ when $A \in \mathbb{Z}^{r \times m}$ is a positive matrix and in the latter, Spiegel extended this to $|S_1(A, \mathbf{b}) \cap [n]_p^m|$ for an arbitrary $\mathbf{b} \in \mathbb{Z}^r$ when A is positive and also abundant.

Proof strategy and further outlook. The core of the common proof of Theorems 4 and 7 is based on a combination of algebraic ideas dealing with the matrix associated with the linear system of equations and the method of moments in probability theory to show convergence towards a normal distribution. The second part is highly inspired by the techniques developed by Ruciński in [9] where he established a similar result to Theorems 4 and 7 for the distribution of the number of occurrences of a fixed graph G as a subgraph in a binomial random graph. However, we want to stress that our work highly differs from [9] on the more difficult linear algebraic structure of the patterns we want to take into account.

Let us now discuss the potential necessity of the sufficient conditions on the probability function p established in our results. In fact, in [9] Ruciński showed that the corresponding conditions in the graph setting were necessary as well as sufficient. In our context we can say something similar regarding the condition that $n(1-p)$ is unbounded. Observe that the expression $n(1-p)$ can be interpreted as the expected number of elements that are *not* chosen in $[n]_p$, hence if $n(1-p) \not\to \infty$, then $[n]_p$ is typically all the interval $[n]$ with the exception of a bounded number of elements. Then it is easy to show that under this condition, X_n is strongly concentrated around its mean value, concluding that $\tilde{X}_n \xrightarrow{d} 0$, and hence, we do not have a normal limiting distribution for the number of solutions.

The second sufficient condition essentially means that $[n]_p$ will contain an infinite number of solutions to the densest part of the given system of linear equations. When stated in its most general form, proving that this is also necessary would require a very delicate and technical analysis, but the following arguments show that at least for the consequences stated in this abstract, Theorems 4 and 7, necessity is very likely.

Let us start with the setting of Theorem 7, that of non-trivial solutions in strictly balanced homogeneous systems of linear equations. Here, as discussed before, Rué, Spiegel and Zumalacárregui in [11] already established a threshold result showing that when $np^{c(A)} = o(1)$, asymptotically almost surely $S_1(A, \mathbf{0}) \cap [n]_p^m$ is empty. Using a concentration argument similar to the one above when discussing the necessity of $n(1-p) \to \infty$, we see that in this case $|S_1(A, \mathbf{0}) \cap [n]_p^m|$ will converge in distribution to the constant 0 distribution. In addition to this threshold results, the authors also studied the distribution at the threshold, that is, the case $np^{c(A)} \to a > 0$ where a is a constant. Their results show that here, the random variable $|S_1(A, \mathbf{0}) \cap [n]_p^m|$ converges in distribution to a Poisson. Putting all of this together, we see that this establishes the only if direction for Theorem 7.

Let us now turn to the setting of Theorem 4, that of proper solutions, where there are no repeated variables by definition. The argument that $np^{c(A)} = o(1)$ implies that $\tilde{X}_n \xrightarrow{d} 0$ is the same as before, so what remains is to understand the case when $np^{c(A)}$ tends to some positive constant. When A is strictly balanced, one can use the same arguments that were used by Rué, Spiegel and Zumalacárregui in the proof of the strictly balanced homogeneous case for the distribution of non-trivial solutions to see that one will also have a Poisson distribution in the setting of Theorem 4. When A is not strictly balanced, an analysis as was performed by Ruciński in the subgraph setting is needed, but since we are now in the situation that all variables are pairwise distinct, the same result should follow, which would establish the necessity of $np^{c(A)} \to \infty$.

To conclude, let us mention that once one has proved a normal limiting distribution, a natural next question is the study of local limit theorems as well as anticoncentration results and tail estimates in a general context. This has been a very active trend of research in the last years, see for instance [19, 3].

References

[1] J. Balogh, R. Morris, and W. Samotij. "Independent sets in hypergraphs". In: *J. Amer. Math. Soc.* 28.3 (2015), pp. 669–709. DOI: 10.1090/S0894-0347-2014-00816-X.

[2] Y. Barhoumi-Andréani, C. Koch, and H. Liu. "Bivariate fluctuations for the number of arithmetic progressions in random sets". In: *Electron. J. Probab.* 24 (2019), Paper No. 145, 32. DOI: 10.1214/19-ejp391.

[3] R. Berkowitz, A. Sah, and M. Sawhney. "Number of arithmetic progressions in dense random subsets of $\mathbb{Z}/n\mathbb{Z}$". In: *Israel J. Math.* 244.2 (2021), pp. 589–620. DOI: 10.1007/s11856-021-2180-7.

[4] D. Conlon and W. T. Gowers. "Combinatorial theorems in sparse random sets". In: *Ann. of Math. (2)* 184.2 (2016), pp. 367–454. DOI: 10.4007/annals.2016.184.2.2.

[5] P. Frankl, R. L. Graham, and V. Rödl. "Quantitative theorems for regular systems of equations". In: *J. Combin. Theory Ser. A* 47.2 (1988), pp. 246–261. DOI: 10.1016/0097-3165(88)90020-9.

[6] R. Hancock, K. Staden, and A. Treglown. "Independent sets in hypergraphs and Ramsey properties of graphs and the integers". In: *SIAM J. Discrete Math.* 33.1 (2019), pp. 153–188. DOI: 10.1137/18M119642X.

[7] Y. Kohayakawa, T. Łuczak, and V. Rödl. "Arithmetic progressions of length three in subsets of a random set". In: *Acta Arith.* 75.2 (1996), pp. 133–163. DOI: 10.4064/aa-75-2-133-163.

[8] K. F. Roth. "On certain sets of integers". In: *J. London Math. Soc.* 28 (1953), pp. 104–109. DOI: 10.1112/jlms/s1-28.1.104.

[9] A. Ruciński. "When are small subgraphs of a random graph normally distributed?" In: *Probab. Theory Related Fields* 78.1 (1988), pp. 1–10. DOI: 10.1007/BF00718031.

[10] J. Rué, O. Serra, and L. Vena. "Counting configuration-free sets in groups". In: *European J. Combin.* 66 (2017), pp. 281–307. DOI: 10.1016/j.ejc.2017.06.027.

[11] J. Rué, C. Spiegel, and A. Zumalacárregui. "Threshold functions and Poisson convergence for systems of equations in random sets". In: *Math. Z.* 288.1-2 (2018), pp. 333–360. DOI: 10.1007/s00209-017-1891-2.

[12] J. Rué and M. Wötzel. "Normal limiting distributions for systems of linear equations in random sets". In: *Linear Algebra Appl.* 649 (2022), pp. 122–151. DOI: 10.1016/j.laa.2022.05.003.

[13] I. Z. Ruzsa. "Solving a linear equation in a set of integers. I". In: *Acta Arith.* 65.3 (1993), pp. 259–282. DOI: 10.4064/aa-65-3-259-282.

[14] D. Saxton and A. Thomason. "Hypergraph containers". In: *Invent. Math.* 201.3 (2015), pp. 925–992. DOI: 10.1007/s00222-014-0562-8.

[15] M. Schacht. "Extremal results for random discrete structures". In: *Ann. of Math. (2)* 184.2 (2016), pp. 333–365. DOI: 10.4007/annals.2016.184.2.1.

[16] C. Spiegel. "A note on sparse supersaturation and extremal results for linear homogeneous systems". In: *Electron. J. Combin.* 24.3 (2017), Paper No. 3.38, 19.

[17] C. Spiegel. "Additive structures and randomness in combinatorics". http://hdl.handle.net/2117/328203. PhD thesis. Universitat Politècnica de Catalunya, 2020.

[18] E. Szemerédi. "On sets of integers containing no k elements in arithmetic progression". In: *Acta Arith.* 27 (1975), pp. 199–245. DOI: 10.4064/aa-27-1-199-245.

[19] L. Warnke. "Upper tails for arithmetic progressions in random subsets". In: *Israel J. Math.* 221.1 (2017), pp. 317–365. DOI: 10.1007/s11856-017-1546-3.

On a nonabelian Kneser theorem

Oriol Serra[1*], Maximilian Wötzel[2†]

[*]Dept. of Mathematics and Institute of Mathematics, UPC, Barcelona
[†]Department of Mathematics, Radboud University, Nijmegen, Netherlands.

Abstract

The well known theorem of Kneser, stating that small sumsets in abelian groups must be periodic, does not hold in nonabelian groups. We prove that in a finite nonabelian group G, a weaker version of Kneser's theorem does hold, stating that if a symmetric set $1 \in S = S^{-1} \subset G$ satisfies $|S^2| < 2|S| - 1$ then S^2 is almost periodic. This is shown in the more general context of vertex transitive graphs. Perhaps surprisingly, the corresponding statement for infinite vertex transitive graphs turns out to be false.

1 Introduction

For two nonempty finite subsets A, B in a group G their *Minkowski product* (or *product set*) is the set $AB = \{ab : a \in A, b \in B\}$. In the sequel, whenever the group G in question is abelian, we will use the additive notation $A + B$ instead and refer to this as the *sumset* or *Minkowski sum*. If G is an abelian torsion-free group or a finite group of prime order [1, 2], one of the basic inequalities in additive combinatorics states either $A + B = G$ or

$$|A + B| \geq |A| + |B| - 1.$$

The above inequality may fail to hold in general abelian groups, but one of the basic theorems due to Kneser [10] provides the structure of pairs of sets for which the inequality does not hold. A set A is *periodic* if there is a nontrivial subgroup $H < G$ such that $A + H = A$. In this case, A is a union of H–cosets. An equivalent formulation of Kneser's theorem is the following.

Theorem 1 ([10])**.** *Let A be a nonempty finite set in an abelian group G. If*

$$|A + B| < |A| + |B| - 1,$$

then $A + B$ is periodic.

Kneser's theorem cannot be extended directly to nonabelian groups. Olson [15] provides examples of pairs of sets A, B in a nonabelian group G such that $|AB| < |A| + |B| - 1$, but for any nontrivial subgroup $H < G$ it holds that $\min\{|HAB|, |AHB|, |ABH|\} > |AB|$. There have been alternative proposals for a nonabelian version of Kneser's theorem. The following is due to Hamidoune [8].

Theorem 2 (Corollary 3.2 in [8])**.** *Let A be a finite nonempty subset of a group G, and denote by G' the subgroup generated by $A^{-1}A$. If*

$$\max\{|A^{-1}A|, |AA^{-1}|\} \leq \min\{2|A| - 2, |G'| - 1\},$$

then there exists a nontrivial subgroup $H < G'$ and an element $a \in A$ such that either $A^{-1}HA = A^{-1}A \cup a^{-1}Ha$ or $AHA^{-1} = AA^{-1} \cup aHa^{-1}$

[1]Email: oriol.serra@upc.edu. Research of O. S. supported by the Spanish Agencia Estatal de Investigación under project PID2020-113082GB-I00.

[2]Email: maximilian.woetzel@ru.nl. Research of M. W. supported by the Dutch Science Council (NWO) under grant number OCENW.M20.009.

In Theorem 2 periodicity with respect to cosets is substituted by periodicity with respect to sets of the form xHy with possibly one exception. Matt DeVos, in an unpublished manuscript, suggests the following generalization of Kneser's theorem.

Theorem 3 ([3]). *Let A, B be subsets of a group G with*

$$|AB| < |A| + |B| - 1.$$

There is a function $f : AB \to G$ such that

$$AB = \bigcup_{x \in AB} xH^{f(x)},$$

where $H^g = g^{-1}Hg$ denotes the conjugate of H by g.

In Theorem 3 periodicity with respect to a subgroup is relaxed to union of cosets of conjugate subgroups.

Extending a theorem of Mann [13] to nonabelian groups, Olson [14] and Hamidoune [8] showed independently that, if $|AB| < |A| + |B| - 1$ then there is a subgroup H satisfying the inequality in the place of one of the two subsets. This in particular can be seen to imply that $|A^2| \geq (3/2)|A|$. Freĭman [4] shows that the equality holds if and only if A is a subgroup. Extending this result, Hamidoune [8] showed that, if $|A^{-1}A| < (5/3)|A|$ then $A^{-1}A$ is periodic by a normal subgroup K and can be covered by at most six cosets of K. Tao [16] partially extended this result by stating that, for every $\epsilon > 0$ if $|A^{-1}A| < (2 - \epsilon)|A|$, then A can be covered with $2/\epsilon$ cosets of a subgroup.

2 Main Results

Our main result shows that periodicity may fail to hold by at most one coset, hence being comparable to Theorem 2. We say that a finite nonempty set A in a group G is *almost right-periodic* if there exists a nontrivial subgroup $H < G$ and an element $a \in A$ such that $AH = A \cup aH$.

Theorem 4. *Let A be a finite nonempty subset of a group G such that $a^{-1}A = A^{-1}a$ for some $a \in A$. If*

$$|A^{-1}A| < 2|A| - 1,$$

then $A^{-1}A$ is almost right-periodic.

Note that the condition on A is for instance satisfied in the special case when A itself is symmetric and contains the neutral element. This symmetry condition on A can be weakened in a way that is better explained by placing the problem in the context of (directed) graphs. The Cayley graph $\Gamma = Cay(G, S)$ has the elements of the group G as its vertices and every element x is adjacent to xs for each $s \in S$. It is a directed graph in general, and undirected whenever $S = S^{-1}$. The action of G on $V(\Gamma)$ by left translations is a transitive automorphism group. Thus a Cayley graph is vertex transitive (of degree $|S|$), although the class of vertex transitive graphs is much wider than the class of Cayley graphs. In this context XS is simply the neighborhood $N_\Gamma(X)$ of X in Γ, and sets with small sumset correspond to sets with small neighborhood in the graph.

Let Γ be a reflexive (that is, containing a loop at every vertex) vertex transitive graph with finite degree $d(\Gamma)$[3]. We call a nonempty finite set $X \subset V(\Gamma)$ a *fragment* if it minimizes the boundary operator $|\partial(X))| = |N_\Gamma(X) \setminus X|$ among finite vertex sets such that $N_\Gamma(X) \neq G$. This minimum value is the *connectivity* $\kappa(\Gamma)$ of the graph. Fragments with smallest cardinality are called *atoms* of Γ, and this size is denoted by $\alpha(\Gamma)$. We have $\kappa(\Gamma) \leq d(\Gamma) - 1$ and there is equality if and only if the atoms are

[3]Note that counter to conventions in other areas of graph theory, we will count the loop at a vertex as only contributing 1 to the degree.

the vertices of the graph. If $\kappa(\Gamma) < d(\Gamma) - 1$ and Γ is directed, it might happen that $\alpha(\Gamma) \neq \alpha(\Gamma^{-1})$. However, note that if $V(\Gamma)$ is finite, it will always hold that $\kappa(\Gamma) = \kappa(\Gamma^{-1})$.

Our proof techniques follow what Hamidoune dubbed the *isoperimetric method*. Original arguments similar to this method were used by Mader [11, 12] purely in the context of vertex-transitive graphs. Hamidoune then further generalized Mader's arguments and used them to prove several results in the setting of groups [5, 6, 7, 8]. A key result of Hamidoune [7] states that, if Γ is reflexive and vertex transitive, $\kappa(\Gamma) < d(\Gamma) - 1$ and $\alpha(\Gamma) \leq \alpha(\Gamma^{-1})$ then the atoms of Γ form a system of imprimitivity of the automorphism group $\mathrm{Aut}(\Gamma)$. In particular, if Γ is a Cayley graph, then the atoms are right-cosets of a nontrivial subgroup. We say that a set $X \subset V(\Gamma)$ is almost periodic if it is the union of blocks of a system of imprimitivity of $\mathrm{Aut}(\Gamma)$ except maybe by precisely one block which is only partially contained in X. In this more general setting Theorem 4 can be extended to the following result.

Theorem 5. *Let Γ be a reflexive finite vertex transitive graph with degree $d(\Gamma)$. If $\kappa(\Gamma) < d(\Gamma) - 1$ and $\alpha(\Gamma) = \alpha(\Gamma^{-1})$ then for every vertex $x \in V(\Gamma)$, the alternated second neighborhood*

$$N(N^{-1}(x))$$

is almost periodic with respect to the atoms of Γ^{-1}.

It is now not too difficult to see that Theorem 4 follows from Theorem 5 by setting $\Gamma = Cay(G, S)$ with $S = a^{-1}A$, noting that $S^{-1}S = A^{-1}A$. In fact, one can make the following slightly stronger statement.

Let G be a finite group and S a finite set with $1 \in S$. If there exists a finite set $X \subset G$ such that

$$|XS| < |X| + |S| - 1,$$

then there is a nontrivial subgroup $H < G$ such that $S^{-1}S$ is almost periodic, provided that the atoms of $Cay(G, S)$ and those of $Cay(G, S^{-1})$ have the same cardinality. Note that as a standalone condition, requiring $1 \in S$ is not restrictive since one can always translate S appropriately without changing the cardinality of $S^{-1}S$. On the other hand, examples of Cayley graphs Γ for which $\alpha(\Gamma) < \alpha(\Gamma^{-1})$ are given for example by Zémor [17], even if the constructions are rather peculiar. A more general approach to construct large families of such groups was proposed by Hamidoune [7].

The significance of Theorem 5 lies in the fact of supplying a nonabelian analog of Kneser theorem, a result which has seen a very wide range of applications in additive combinatorics. Furthermore, we are able to show that it is optimal in two ways. The first concerns the natural question of whether one can improve almost periodicity to full periodicity in the conclusion of the theorem. We are able to show that this is not possible, even in the more restrictive setting of Cayley graphs.

Theorem 6. *There exists a finite group G and a generating set S of G such that $\Gamma = Cay(G, S)$ satisfies all the conditions of Theorem 5, but*

$$\min\{|HS^{-1}S|, |S^{-1}HS|, |S^{-1}SH|\} > |S^{-1}S|$$

for every nontrivial subgroup $H < G$.

In this way, the construction given by Theorem 6 is a strengthening of the previously mentioned examples given by Olson [15] to the stricter structure of AA^{-1} instead of AB.

Another natural question to ask is whether Theorem 5 also holds in infinite graphs. Again, we are able to construct a family of examples of infinite vertex-transitive graphs of counter-examples. Specifically, we show the following.

Theorem 7. *Let $n \in \mathbb{N}$ be a positive integer. Then there exists an infinite, undirected, reflexive, vertex-transitive graph Γ_n such that every vertex $x \in V(\Gamma_n)$ satisfies*

$$|H| > |N(N^{-1}(x)) \cap H| > 0$$

for $2n - 1$ distinct atoms H.

Proof. For clarity of presentation, we will restrict ourselves for now to the case $n = 1$ and mention at the end how to generalize this to arbitrary n. Let $F_2 = \langle a, b\rangle$ be the free group of rank two and consider the undirected Cayley graph

$$\Gamma = Cay(F_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4, S),$$

where $S = \{1, a, b, a^{-1}, b^{-1}\} \times \mathbb{Z}_4 \times \mathbb{Z}_4$. This can be visualized as the infinite 4–regular tree where each vertex is replaced by a copy of the complete graph K_8. For every $w \in F_2$, the sets $\{w\} \times \mathbb{Z}_4 \times \mathbb{Z}_4$ are clearly blocks of the automorphism group of the graph defined by the left translations of the group $G = F_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$.

Let $\pi : \mathbb{Z}_4 \to \{a, b, a^{-1}, b^{-1}\}$ be a bijection and let $H < F_2$ be the subgroup of words in F_2 of even reduced length (those elements in F_2 at even distance from 1 in $Cay(F_2, \{a, b, a^{-1}, b^{-1}\})$). Consider the directed graph $\vec{\Gamma}_0$ with vertex set $V(\Gamma)$ whose directed edges are defined by

$$N^+_{\vec{\Gamma}_0}(w, i, j) = \{w\pi(j)\} \times (i + 1_H(w) + \{0, -1\}) \times \mathbb{Z}_4 = \begin{cases} \{w\pi(j)\} \times \{i, i-1\} \times \mathbb{Z}_4, & \text{if } w \in H; \\ \{w\pi(j)\} \times \{i, i+1\} \times \mathbb{Z}_4, & \text{if } w \notin H. \end{cases}$$

The directed graph $\vec{\Gamma}_0$ is regular with $d^+(\vec{\Gamma}_0) = d^-(\vec{\Gamma}_0) = 8$. Figure 1 illustrates the out- and in-neighborhoods of $\{1\} \times \mathbb{Z}_4 \times \mathbb{Z}_4$ for the bijection $\pi(0, 1, 2, 3) = (a, b, a^{-1}, b^{-1})$.

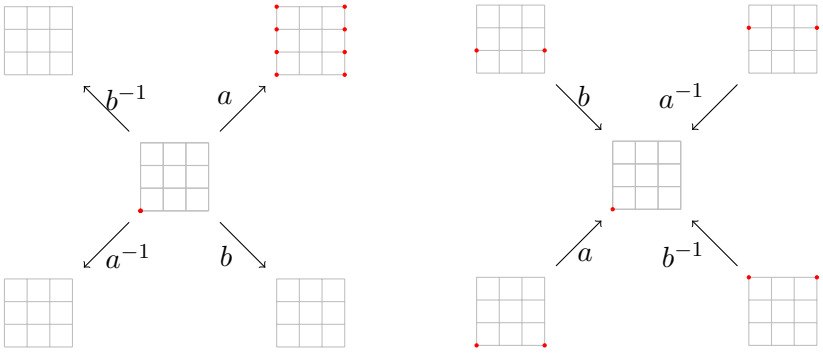


Figure 1: The out- and in-neighborhoods of $(1, 0, 0)$ in $\vec{\Gamma}_0$.

For every $g = (u, \alpha, 0) \in G$ the map $\phi_g : G \to G$ defined by

$$\phi_g(w, i, j) = (uw, i + \alpha + 1_H(u)1_H(w), j).$$

satisfies

$$\begin{aligned} \phi_g(w\pi(j), i + 1_H(w), j') &= (uw\pi(j), i + \alpha + 1_H(uw) + 1_H(u)1_H(w\pi_j), j') \\ &= (uw\pi(j), i + \alpha + 1_H(w\pi(j)1_H(u)1_H(w)), \end{aligned}$$

so that $\phi_g(N_{\vec{\Gamma}_0}(w, i, j)) = N_{\vec{\Gamma}_0}(\phi_g(w, i, j))$ and ϕ_g is an automorphism of $\vec{\Gamma}_0$ sending $(1, 0, 0)$ to g. On the other hand, for $g = (1, 0, \beta)$, the map $\psi_g : G \to G$ defined as $\psi_g(w, i, j) = (f_{\beta,j}(w), i, j + \beta)$, where $f_{\beta,j}(w) = w$ if $\pi(j + \beta) = \pi(j)$ and $f_{\beta,j}$ exchanges a and b in w otherwise, is an automorphism that maps $(1, 0, 0)$ to $(1, 0, j)$. It follows that $\vec{\Gamma}_0$ is vertex transitive.

Let Γ_0 be the undirected simple graph obtained by ignoring the directions of the arcs in $\vec{\Gamma}_0$. We observe that Γ_0 is a vertex transitive subgraph of Γ with degree $d(\Gamma_0) = 14$.

Let Γ_1, the spanning subgraph of Γ with edge set $E(\Gamma_1) = E(\Gamma) \setminus E(\Gamma_0)$. By the above remarks, Γ_1 is reflexive and vertex transitive of degree $d(\Gamma_1) = d(\Gamma) - d(\Gamma_0) = 66$. The boundary of $\{1\} \times \mathbb{Z}_4 \times \mathbb{Z}_4$ has size 64 (the four blocks $\{gs\} \times \mathbb{Z}_4 \times \mathbb{Z}_4$ for $s \in \{a, b, a^{-1}, b^{-1}\}$) so that $\kappa(\Gamma_1) < d(\Gamma_1) - 1$. One can easily check that no other set has smaller boundary (because of the exponential expansion of the quotient graph $Cay(F_2, \{1, a, b, a^{-1}, b^{-1}\})$ and the local structure of the complete graphs K_8), so that

these blocks are atoms of Γ_1. However, the second neighborhood of $(1,0,0)$ is not almost periodic by the imprimitivity system of atoms. In fact, $N(N((1,0,0)))$ has holes in precisely 3 blocks.

The above example can be generalized by starting with the free group F_{n+1} of rank $n+1$ and considering the corresponding group $F_{n+1} \times \mathbb{Z}_4 \times \mathbb{Z}_{2n+2}$. In the resulting graph the second neighborhood of a vertex has holes in $2n-1$ atoms. □

The question remains whether the graphs Γ_n are Cayley graphs, and if not, whether one can find similar constructions which are. As a remark, note that only the second neighborhood of a vertex $x \in V(\Gamma_n)$ avoids almost periodicity. In fact, the third neighborhood (and hence any following one) is fully periodic with respect to the atoms.

3 Final Remarks

The almost periodic structure of sets S in a group with small value of $|S^{-1}S|$ seems to us a nice extension of Kneser's theorem to nonabelian groups. The fact that it can be phrased in the more general context of vertex transitive graphs enhances the combinatorial nature of the result. It is worth noting that the notion of almost periodicity already occurs in the abelian case in the characterization of critical pairs, that is pairs (A,B) satisfying the equality $|A+B| = |A| + |B| - 1$, obtained by Kemperman [9].

There is a number of natural questions arising from the result.

1. The fact that atoms of a vertex transitive graph Γ can have different size compared to the ones of Γ^{-1} is intriguing. In the finite case, it may happen that atoms of Γ or Γ^{-1} may cease to be a system of imprimitivity of the automorphism group, a crucial property in our arguments (for infinite locally finite graphs this property holds in both Γ and Γ^{-1}). A better understanding of these graphs may shed some light on this somewhat stringent assumption in Theorem 5.

2. In the study of so–called Plünecke inequalities, an instrumental tool to settle Babai's conjecture of expansion in simple groups, it was already noticed that third neighborhoods rather than second neighborhoods where the appropriate object to look at. It might be worth analyzing if Kneser's theorem in nonabelian groups can be more appropriately stated in terms of larger iterations of neighborhoods of a vertices, as the examples we have encountered suggest.

3. An important feature of Kneser's theorem is its asymmetric nature, applying to arbitrary pairs of sets. In additive combinatorics it is customary to analyze single sets and their doubling, but asymmetric versions of the theorem are certainly worth exploring.

References

[1] A. L. Cauchy. "Recherches sur les nombres". In: *J. École Polytech* 9 (1813), pp. 99–116.

[2] H. Davenport. "On the Addition of Residue Classes". In: *Journal of the London Mathematical Society* s1-10.1 (1935), pp. 30–32. DOI: https://doi.org/10.1112/jlms/s1-10.37.30.

[3] M. DeVos. *The Structure of Critical Product Sets*. Preprint. 2013. DOI: 10.48550/ARXIV.1301.0096.

[4] G. A. Freĭman. "Groups and the inverse problems of additive number theory". In: *Number-theoretic studies in the Markov spectrum and in the structural theory of set addition*. 1973, pp. 175–183.

[5] Y. O. Hamidoune. "Sur les atomes d'un graphe orienté". In: *C. R. Acad. Sci. Paris Sér. A-B* 284.20 (1977), A1253–A1256.

[6] Y. O. Hamidoune. "Quelques problèmes de connexité dans les graphes orientés". In: *J. Combin. Theory Ser. B* 30.1 (1981), pp. 1–10. DOI: 10.1016/0095-8956(81)90085-X.

[7] Y. O. Hamidoune. “Some additive applications of the isoperimetric approach”. In: *Ann. Inst. Fourier (Grenoble)* 58.6 (2008), pp. 2007–2036.

[8] Y. O. Hamidoune. “Two inverse results”. In: *Combinatorica* 33.2 (2013), pp. 217–230. DOI: 10.1007/s00493-013-2731-6.

[9] J. H. B. Kemperman. “On small sumsets in an abelian group”. In: *Acta Math.* 103 (1960), pp. 63–88. DOI: 10.1007/BF02546525.

[10] M. Kneser. “Abschätzung der asymptotischen Dichte von Summenmengen”. In: *Math. Z.* 58 (1953), pp. 459–484. DOI: 10.1007/BF01174162.

[11] W. Mader. “Über den Zusammenhang symmetrischer Graphen”. In: *Arch. Math. (Basel)* 21 (1970), pp. 331–336. DOI: 10.1007/BF01220924.

[12] W. Mader. “Eine Eigenschaft der Atome endlicher Graphen”. In: *Arch. Math. (Basel)* 22 (1971), pp. 333–336. DOI: 10.1007/BF01222585.

[13] H. B. Mann. “An addition theorem of Abelian groups for sets of elements”. In: *Proc. Amer. Math. Soc.* 4 (1953), p. 423. DOI: 10.2307/2032143.

[14] J. E. Olson. “On the sum of two sets in a group”. In: *J. Number Theory* 18.1 (1984), pp. 110–120. DOI: 10.1016/0022-314X(84)90047-7.

[15] J. E. Olson. “On the symmetric difference of two sets in a group”. In: *European J. Combin.* 7.1 (1986), pp. 43–54. DOI: 10.1016/S0195-6698(86)80016-6.

[16] T. Tao. “Noncommutative sets of small doubling”. In: *European J. Combin.* 34.8 (2013), pp. 1459–1465. DOI: 10.1016/j.ejc.2013.05.028.

[17] G. Zémor. “On positive and negative atoms of Cayley digraphs”. In: *Discrete Appl. Math.* 23.2 (1989), pp. 193–195. DOI: 10.1016/0166-218X(89)90027-9.

A faster algorithm for the two cluster partitioning problem

Francisco-Javier Soto[1*], Ana I. Gómez[2†], Domingo Gómez-Pérez[3*]

[*]Facultad de Ciencias, Universidad de Cantabria
[†]Esc. Tec. Sup. de Ingeniería Informática, Universidad Rey Juan Carlos

A software implementation of the proposal can be found in `https://github.com/domingoUnican/Mejora_k_means`. *This research is supported by the grant PID2019-110633GB-I00 funded by MCIN/ AEI /10.13039/501100011033*

Abstract

Clustering (partitioning a set into different subsets) are among the most widely used type of algorithms for unstructured data and often means solving a combinatorial problem of finding the minimum of an objective function. The most popular clustering algorithm in practice is Lloyd's heuristic approximation algorithm to the k-means optimum centroids. A crucial issue of this approach is that there is only a probabilistic guarantee regarding the goodness of the solution. Indeed, it is possible to construct datasets where Lloyd's heuristic approximation algorithm only finds local minimums. A challenging problem is finding the global optimum for any given dataset. Although this problem is NP-hard, new algorithms for finding optimal clustering offer many benefits such as the research of new heuristic algorithms. This work analyses possible improvements for solving this problem, focusing on optimizing the global searches with branch-and-bound techniques. The numerical results show a promising computational advantage for the case of the partitioning of two sets over previous proposals.

1 Introduction

Clustering problems have attracted a lot of attention in many different areas related to algorithms that deal with unstructured data, for example in data mining. Although the fitness of a clustering is context-dependent, it is quite common to reformulate to a combinatorial problem as finding the minimum of an objective function. The most widely popular choice is to represent the data as points in a real d-dimensional space and take the objective function as the mean squared distance from each data point to the mass center of the cluster it belongs. Unfortunately, it is known that the formulation of the problem is NP-Hard [5], even in the case that $k = 2$ [3], fostering the development of probabilistic approximation algorithms based on Lloyd's heuristic algorithm [1, 7, 8].

A key problem related to probabilistic algorithms is at conceptual level: they do not offer deterministic bounds and it is not possible to find the real optimum even in small to moderate datasets. Additionally, there are room for improvements in the analysis because most of the algorithms perform better than expected in real world applications. Hence it is necessary to conduct numerical experiments and comparing the output of the algorithms with the real optimum, and that requires fast algorithms for finding the optimum.

Brute force search, i. e. evaluating all partitions, is computationally possible for very small datasets, i.e. less than 30 points [10]. The first results into more efficient searches [6] found that finding the optimum can be done in polynomial time on the number of points if the dimension and k are fixed.

[1]Email: francisco-javier.soto@alumnos.unican.es
[2]Email: ana.gomez.perez@urjc.es
[3]Email: gomezd@unican.es

Numerical experiments on finding the optimum were conducted in a more recent work [10], improving the complexity of the original algorithm. Although this implementation improved all known proposals, it requires the use of super-computation for datasets with more than 300 points, each of them with four numerical attributes attached. This opens new research lines for improvements to work efficiently on larger datasets.

The aim of our research is to extend the algorithm by Avis and Fukuda [2], in order to adapt it to the branch-and-bound paradigm with the use of a new heuristic generalizing the findings of Wang [11].

The paper is organized as follows: Section 2 explains the algorithm for cell enumeration given by Avis and Fukuda [2]. In Section 3 we recall the relation between finding optimal clustering for $k = 2$ and cell enumeration. Finally, section 4 evaluates this new algorithm with simulations and finishes with the conclusions.

2 Hyperplanes Arrangements and the CellEnum Algorithm

We denote by $\vec{a} = (a_1, \ldots, a_d)$ elements of the vector space of dimension d, $\mathbb{R}^d$, where d is a positive integer and $\mathbb{R}$ the field of real numbers. We also consider matrices and their usual operations, namely multiplication, addition and transposition. The notation for the transposition of a matrix $\mathbf{A}$ is $(\mathbf{A})^{\mathbf{T}}$. Then, vectors in $\mathbb{R}^d$ are considered as matrices with d rows and 1 column. We also use $\vec{a} \cdot \vec{x}$ to denote $(\vec{a})^{\mathbf{T}}\vec{x}$, which correspond to the standard dot product, and the matrix form $\mathbf{A}\vec{x} = \vec{b}$ to encode the finite set of hyperplanes $\mathcal{H} = \{\mathcal{H}_1, \ldots, \mathcal{H}_m\}$. Let $\mathbf{A}_{i,\cdot}$ denotes the ith row of matrix $\mathbf{A}$, then

$$\mathcal{H}_i = \{\vec{x} \in \mathbb{R}^d \mid \mathbf{A}_{i,\cdot}\vec{x} = b_i\}, \quad \vec{b} = (b_1, \ldots, b_m). \tag{1}$$

Definition 1. A set of hyperplanes in $\mathbb{R}^d$ partitions the space into relatively open convex polyhedral regions, called faces. This partition is called a hyperplane arrangement.

We make a distinction between the two sides of a hyperplane. A point $\vec{p} \in \mathbb{R}^d$ is on the positive side of hyperplane $\mathcal{H}_i$, denoted by $\mathcal{H}_i^+$, if $\mathbf{A}_{i,\cdot}\vec{p} > b_i$. Otherwise, it can be on $\mathcal{H}_i$ or in the negative side denoted by $\mathcal{H}_i^-$. Therefore, for $\vec{p} \in \mathbb{R}^d$ we define a sign vector of length m consisting of $+, 0, -$ signs as follows:

$$sv(\vec{p})_i = \begin{cases} + & \text{if } \vec{p} \in \mathcal{H}_i^+, \\ - & \text{if } \vec{p} \in \mathcal{H}_i^-, \\ 0 & \text{if } \vec{p} \in \mathcal{H}_i, \end{cases}$$

where $i = 1, \ldots, m$ and m is the number of hyperplanes.

Definition 2. A face is a set of points with the same sign vector. It is called a k-face if its dimension is $k \leq d$ and a cell if the dimension is d. The support of a face f is the set of indexes of the non-zero elements in the sign vector. The positive (resp. negative) support of f is the set of indices of the hyperplanes of positive (resp. negative) sign and it is denoted by f^+ (resp. f^-).

Consider the state space of all possible values of $sv(\vec{p})$ that could possibly exist in an hyperplane arrangement. The state space consists of at most $O(m^d)$ sign vectors [4, page 8]. Given that they can be generated efficiently [9], i.e. it is possible to explore all states doing $O(m)$ operations per state, where the implied constant depends on d. The algorithm CellEnum [2] do this by assigning to each cell (except the root cell) a unique parent cell, thereby obtaining a tree structure for the cells and traverses it from the root to the leaves. One procedure, called (ParentSearch) discovers the parent cell of any given cell, which is a chosen neighbor that has one more $+$ in its sign vector. Another key procedure, the adjacency oracle (AllAdj), returns all neighbours of a cell, i.e. any cell with a sign vector with a additional $-$ sign.

Without loss of generality, the sign vector f of the root cell consists only of $+'s$, multiplying by -1 all hyperplanes in f^-.

3 Improvements on finding a real Optimum

For a set S of m points $\vec{p_i} \in \mathbb{R}^d$, we denote by $|S|$ the number of points of S and $\vec{x}(S)$ its mass center.

Definition 3. The general $k-$clustering is a partition of a given set $S \subset \mathbb{R}^d$ of m points $\vec{p_i}$, $i = 1, \ldots, m$ into k disjoint nonempty subsets $S_1, \ldots, S_k$ called clusters. A variance-based $k-$clustering problem is to find k clusters such that the following holds,

$$\min_{S_1,\ldots,S_k} \mathrm{Var}(S_1, \ldots, S_k) = \min_{S_1,\ldots,S_k} \left(\sum_{j=1}^{k} \frac{1}{|S_j|} \sum_{\vec{p_i} \in S_j} \|\vec{p_i} - \vec{x}(S_j)\|^2 \right).$$

For the rest of the paper, we consider only the case of $k = 2$.

A possible method to partition S is using the concept of Voronoi diagrams. The idea behind it is to pick 2 points, called the centers, $\vec{q_1}, \vec{q_2}$ and divide S into subsets S_1, S_2 where each of them is defined by the points that are closest to the correspondent center.

Definition 4. Given $\vec{q_1}, \vec{q_2} \in \mathbb{R}^d$ we define $\mathrm{Vor}(\vec{q_1}) = \{\vec{p} \in \mathbb{R}^d : \|\vec{p} - \vec{q_1}\|^2 < \|\vec{p} - \vec{q_2}\|^2\}$. A Voronoi partition S_1, S_2 with centers $\vec{q_1}, \vec{q_2}$ is defined as $\vec{p} \in S_1 \iff \vec{p} \in \mathrm{Vor}(\vec{q_1})$.

The Voronoi partitions provide clusters of S that are reasonable candidates to define an optimal partition that minimizes the quantity of Definition 3. Specifically, the real optimum is a Voronoi partition, see [10, Fact 3]. Suppose that a Voronoi partition $\{S_1, S_2\}$ is the solution to the variance-based 2-clustering problem, so, $\vec{p} \in S_1$ if and only if $\|\vec{p} - \vec{q_1}\|^2 < \|\vec{p} - \vec{q_2}\|^2$, which is equivalent to $\vec{p} \cdot (\vec{q_2} - \vec{q_1}) < (||\vec{q_2}||^2 - ||\vec{q_1}||^2)/2$, or

$$\begin{cases} \vec{p} \cdot \frac{2(\vec{q_2} - \vec{q_1})}{||\vec{q_2}||^2 - ||\vec{q_1}||^2} < 1 & \text{if } ||\vec{q_1}||^2 \neq ||\vec{q_2}||^2, \\ \vec{p} \cdot (\vec{q_2} - \vec{q_1}) < 0 & \text{otherwise.} \end{cases} \tag{2}$$

Therefore the state space for the candidate partitions to be optimal in the sense of minimizing the function of Definition 3 can be reduced to enumerate the vector signs of two hyperplanes arrangement defined by the same matrix $\mathbf{A}$ with m rows and the vectors $\vec{b_1}, \vec{b_2}$,

$$\mathbf{A}_{i,\cdot} = \vec{p_i}, \quad i = 1, \ldots, m, \qquad \vec{b_1} = (1, \ldots, 1), \qquad \vec{b_2} = (0, \ldots, 0).$$

We propose the following algorithm 1 based-on-branch and bound method which extends the CellEnumeration Algorithm [2].

Algorithm 1 Algorithm CellEnumBB

Require: cell c of an hyperplane arrangement $\mathcal{H}$, cost() function, bound() function, best-value $\alpha = \infty$
Ensure: all cells in the subtree rooted at c with Branch & Bound technique

```
save c
for all h ∈ AllAdj(c) do
    if h ∈ c+ then
        e = c, e_h=-
        if ParentSearch(e)==h and bound(e) < α then
            α = CellEnumBB(e, min(α, cost(e)))
        end if
    end if
end for
return α
```

There are some details of the implementation of the algorithm above that are necessary to explain. The procedure ParentSearch returns the position of the vector which is different between the given cell and

the parent cell. The procedure *AllAdj* returns a list of the indices of the non-redundant hyperplanes that bound a cell, which is enough to identify the neighbours. Therefore, we can find the optimal partition represented by a sign vector v_0 by applying Algorithm 1 to the hyperplane arrangements described in the Equation 2 and returning v_0 such that $\text{cost}(v_0)$ is the minimum possible.

In order to avoid exploring possible nodes of the tree, the branch-and-bound paradigm requires

- an initial value for best-value which can be possible optimum. This can be found using a probabilistic algorithm,
- an heuristic, bound(e), about the minimum cost of each state in the branch of the tree. Our proposal is to sum the values of the variance-based 2-clustering problem for different orthonormal projections for $d = 1$. An efficient algorithm exists for this case [11].

The idea of projecting is formalized in the following way. Let $\mathcal{O}(n)$ denote the set of orthogonal matrices $\mathbf{Q}$, i.e. satisfying $\mathbf{Q}(\mathbf{Q})^{\mathbf{T}} = \mathbf{I}$. For each matrix $\mathbf{Q}$, clusteringBound($\mathbf{Q}$) is the sum of the variance-based 2-clustering problems given by the projections in the directions defined by the rows. Notice that for $\vec{c}$ fix the membership of the corresponding points. Then, the family of heuristics where the best bound function lies should be

$$B = \sup_{\mathbf{Q} \in \mathcal{O}(n)} \text{clusteringBound}(\mathbf{Q}).$$

Finding the best bound is out of scope of this paper because it can be computationally expensive. We propose and compare two different approaches: use the Principal Component Analysis (PCA) method and take $\mathbf{Q}$ random. The first approach is to consider the PCA matrix, which is a particular orthogonal matrix that has *a priori* good properties. First, let us define the symmetric matrix

$$\mathcal{M} = \sum_{\vec{p} \in S_1} (\vec{p} - \vec{x}(S_1)) \cdot (\vec{p} - \vec{x}(S_1))^T + \sum_{\vec{p} \in S_2} (\vec{p} - \vec{x}(S_2)) \cdot (\vec{p} - \vec{x}(S_2))^T.$$

We consider the orthogonal matrix Γ such that $\Gamma^T \mathcal{M} \Gamma$ is diagonal and $\Gamma_{\cdot,i}$ is the eigenvector associated to the eigenvalue λ_i satisfying $\lambda_1 \geq, ..., \geq \lambda_d$. The idea here is that the first eigenvalues, in particular $\lambda_{max} = \lambda_1$, seem to be potential candidates to explain why one should not explore that part of the search tree, given that the rate $\left(\sum_{i=1}^{l} \lambda_i\right) / \left(\sum_{i=1}^{d} \lambda_i\right)$, $1 \leq l \leq d$, is similar to the proportion of the variance in the technique of PCA. An additional benefit is that the cost() function of Algorithm 1 can be computed easily as $trace(\mathcal{M}) = \sum_{i=1}^{d} \lambda_i$.

Although it is a natural choice, our preliminary computer experiments show that matrix Γ is far from being an optimal choice. The other alternative is to consider a Monte Carlo approach by sampling random matrices from $\mathcal{O}(n)$. Both methods are compared in the next section.

Algorithm 2 Algorithm oneDimCluster

Require: Sets of points $X, Y \subset \mathbb{R}$ where $\max(X) < \min(Y)$
Ensure: Optimal value $Var(S_1, S_2)$ where $\{S_1, S_2\}$ is a partition of $S = X \cup Y$ such that $Y \subset S_2$.
  $S_1 = \emptyset$, $S_2 = X \cup Y$
  $\mu_1 = 0$, $\mu_2 = \vec{x}(S_2)$
  $V = Var(S_1, S_2)$
  for all $i \in \{1, ..., n\}$, $x_i \in X$, $x_i \leq x_{i+1}$ **do**
    $S_1 = S_1 \cup \{x_i\}$, $S_2 = S_2 - \{x_i\}$
    $\mu_1 = \vec{x}(S_2)$, $\mu_2 = \vec{x}(S_2)$
    $V = \min(V, Var(S_1, S_2))$
  end for
  return V

4 Numerical Experiments and Conclusions

We have made preliminary numerical experiments comparing with a previous algorithm developed in 2018 [10] using their dataset for comparison. We use a Macbook Pro with a procesor 2,5 GHz Intel Core i5 and 16 GB RAM without multi-threading techniques. As it can be seen in Figure 1, our heuristics performs worse for $d = 2$ but computational times for $d = 3$ show a clear improvement for both heuristics over the original algorithm and considerably better for the Monte Carlo approach. It is worth to notice that adding points does not automatically increase computation time using the heuristics. We leave for future work to improve this heuristic, include multi-threading techniques and perform more computational experiments in higher dimensions for other datasets.

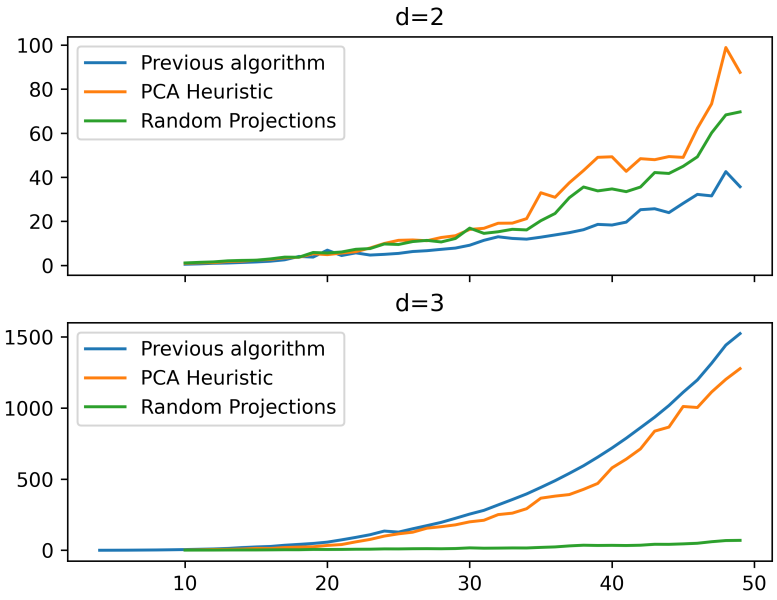


Figure 1: Computational times for $d = 2$ and $d = 3$, comparing previous implementation and two heuristic. The x-axis represents the number of points and the y-axis the computational time in seconds.

References

[1] D. Arthur and S. Vassilvitskii. “k-means++: the advantages of careful seeding”. In: *Proceedings of the eighteenth annual ACM-SIAM symposium on Discrete algorithms*. SODA ’07. New Orleans, Louisiana: Society for Industrial and Applied Mathematics, 2007, pp. 1027–1035.

[2] D. Avis and K. Fukuda. “Reverse Search for Enumeration”. In: *Discrete Applied Mathematics* 65 (1993), pp. 21–46.

[3] P. Drineas, A. M. Frieze, R. Kannan, S. S. Vempala, and V. Vinay. “Clustering Large Graphs via the Singular Value Decomposition”. In: *Mach. Learn.* 56.1-3 (2004), pp. 9–33. DOI: 10.1023/B:MACH.0000033113.59016.96.

[4] H. Edelsbrunner. *Algorithms in combinatorial geometry*. New York, NY, USA: Springer-Verlag New York, Inc., 1987.

[5] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.

[6] M. Inaba, N. Katoh, and H. Imai. “Applications of Weighted Voronoi Diagrams and Randomization to Variance-Based *k*-Clustering (Extended Abstract)”. In: *Symposium on Computational Geometry*. 1994, pp. 332–339.

[7] T. Kanungo, D. M. Mount, N. S. Netanyahu, C. D. Piatko, R. Silverman, and A. Y. Wu. “An Efficient k-Means Clustering Algorithm: Analysis and Implementation”. In: *IEEE Trans. Pattern Anal. Mach. Intell.* 24.7 (2002), pp. 881–892. DOI: 10.1109/TPAMI.2002.1017616.

[8] R. Ostrovsky, Y. Rabani, L. J. Schulman, and C. Swamy. “The effectiveness of lloyd-type methods for the k-means problem”. In: *J. ACM* 59.6 (2012), 28:1–28:22. DOI: 10.1145/2395116.2395117.

[9] N. H. Sleumer. “Hyperplane arrangements: Constructions, Visualizations and Applications”. MA thesis. Zurich: Swiss Federal Institute of Technology, 2000.

[10] C. Tîrnăucă, D. Gómez-Pérez, J. L. Balcázar, and J. L. Montaña. “Global optimality in k-means clustering”. In: *Information Sciences* 439-440 (2018), pp. 79–94. DOI: https://doi.org/10.1016/j.ins.2018.02.001.

[11] H. Wang and M. Song. “Ckmeans.1d.dp: Optimal *k*-means clustering in one dimension by dynamic programming”. In: *The R Journal* 3.2 (2011), pp. 29–33. DOI: 10.32614/RJ-2011-015.

Poster Presentations

Cospectral generalized Johnson and Grassmann graphs

Aida Abiad[1], Jozefien D'haeseleer[2], Robin Simoens[3]

Dept. of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Belgium

This is part of a MSc thesis in pure mathematics. The full version of this work will be published elsewhere.

Abstract

Graphs with the same spectrum are called *cospectral*. A graph is *determined by its spectrum* if every graph cospectral with it is isomorphic to it. We ponder the question

Which graphs are determined by their spectrum?

for the (union of) graphs in the *Johnson scheme* and *Grassmann scheme*. We present several recent results concerning the cospectrality of these graphs, as well as some new discoveries on the cospectrality of *generalized Johnson graphs* and *q-Kneser graphs*.

1 Introduction

An important open problem in spectral graph theory is to find out which graphs are determined by their spectrum (the eigenvalues of their adjacency matrix). It has been conjectured by van Dam and Haemers that almost all graphs are determined by their spectrum [4]. This conjecture, which plays a special role in the graph isomorphism problem, has been solved for several families of graphs; sometimes by proving that the spectrum determines the graph, and sometimes by constructing nonisomorphic graphs with the same spectrum. Several of these methods for constructing nonisomorphic cospectral graphs include: Seidel switching [12], Godsil-McKay switching [8], Wang-Qiu-Hu switching [13], Abiad-Haemers switching [1], techniques based on point-line geometries and GTPT-switching [7]. Most of these techniques work particularly well on graphs coming from finite geometries. In this work we will report recent results concerning the conjecture on generalizations of the Johnson graphs.

2 Preliminaries

A *graph* is a tuple $\Gamma = (V, E)$ with V a finite set of elements, called *vertices*, and E a set of pairs $\{u, v\}$ with $u, v \in V$, $u \neq v$, called *edges*. Two vertices u and v are *adjacent* (notation: $u \sim v$) if $\{u, v\} \in E$. Adjacent vertices are called *neighbours*. The *degree* $\deg(v)$ of a vertex v is the number of its neighbours.

Let Γ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of Γ is the matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ where

$$a_{ij} = \begin{cases} 0 & \text{if } v_i \not\sim v_j \\ 1 & \text{if } v_i \sim v_j \end{cases}.$$

The *spectrum* of Γ is the multiset of all eigenvalues of A. Graphs are *cospectral* if they have the same spectrum. Two graphs that are cospectral but not isomorphic, are called *cospectral mates*. A graph is *determined by its spectrum* (DS) if it has no cospectral mate. on the other hand if a graph has a cospectral mate, it is not determined by its spectrum (NDS).

[1]Email: Aida.Abiad@UGent.be
[2]Email: Jozefien.Dhaeseleer@UGent.be
[3]Email: Robin.Simoens@UGent.be

One of the most successful methods for finding cospectral mates is Godsil-McKay switching. We mainly focus on the following simplified version.

Theorem 1 ([8]). *Let Γ be a graph and C a subset of its vertices such that the following hold:*

(i) The induced subgraph on C is regular.

(ii) Every vertex outside C has exactly 0, $\frac{1}{2}|C|$ or $|C|$ neighbours in C.

For every $u \in C$ and every $v \notin C$ that has exactly $\frac{1}{2}|C|$ neighbours in C, reverse the adjacency between u and v. Then the resulting graph is cospectral with Γ. We say it is obtained from Γ by GM-switching *about C.*

Let n, k be positive integers with $k \leq n$. We now define the (union of) graphs of the Johnson scheme.

Definition 2. Let $S \subseteq \{0, 1, \ldots, k-1\}$. The *generalized Johnson graph* $J_S(n, k)$ has as vertices the k-subsets of $\{1, \ldots, n\}$, where two vertices are adjacent if their intersection size is in S.

In particular, $J_{\{0\}}(n, k)$ is the *Kneser graph* $K(n, k)$ and $J_{\{k-1\}}(n, k)$ is the *Johnson graph* $J(n, k)$.

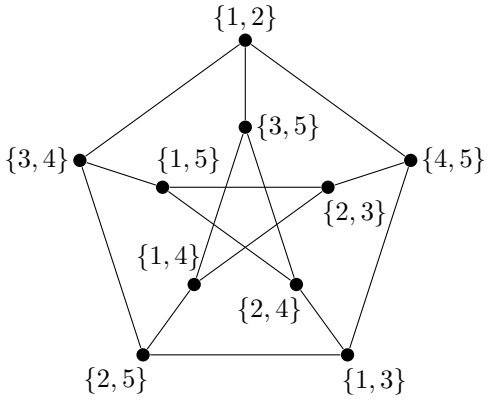


Figure 1: The Petersen graph, represented as the Kneser graph $K(5, 2)$.

Let $\mathbb{F}_q$ be the finite field of order q. Then q is a prime power, i.e. $q = p^h$ with p prime and $h > 0$. A vector space of dimension k is called a *k-space* for short.

Definition 3. Let $S \subseteq \{0, 1, \ldots, k-1\}$. The *generalized Grassmann graph* $J_{q,S}(n, k)$ has as vertices the k-subspaces of $\mathbb{F}_q^n$, where two vertices are adjacent if their intersection dimension is in S.

In particular, $J_{q,\{0\}}(n, k)$ is the *q-Kneser graph* $K_q(n, k)$ and $J_{q,\{k-1\}}(n, k)$ is the *Grassmann graph* $J_q(n, k)$.

Note that we may assume that $k \leq \frac{n}{2}$ since $J_S(n, k)$ is isomorphic to $J_{\{s+n-2k|s\in S\}}(n, n-k)$, and similarly, $J_{q,S}(n, k) \cong J_{q,\{s+n-2k|s\in S\}}(n, n-k)$. We will also assume that $|S| \leq \frac{n}{2}$ because the complement of $J_S(n, k)$ is $J_{\{1,\ldots,n\}\setminus S}(n, k)$, and a regular graph is DS if and only its complement is. Similar for $J_{q,S}(n, k)$. Also note that if $S = \emptyset$, the graph is complete or edgeless and therefore trivially DS.

3 Theoretical results

The following results about the cospectrality of generalized Johnson and Grassmann graphs have been obtained by various authors.

Let q be a prime power and $1 \leq k \leq \frac{n}{2}$.

Theorem 4 ([2, 10]). *$K(n,2)$ is determined by its spectrum if and only if $n \neq 8$.*

Theorem 5 ([11]). *$K(2k+1,k)$ is determined by its spectrum.*

Theorem 6 ([6]). *$J_q(2k+1,k)$ is not determined by its spectrum.*

Theorem 7 ([5]). *$J(n,k)$ and $J_q(n,k)$ are not determined by their spectrum if $3 \leq k$.*

Theorem 8 ([9]). *$K(n,k)$ is not determined by its spectrum if there is an $m < k-1$ such that $\binom{n-k+m}{m} = 2\binom{n-2k+m}{m}$.*

Theorem 9 ([3]). *$J_{\{0,1,\ldots,m\}}(3k-2m-1,k)$ is not determined by its spectrum if $k \geq \max(m+2,3)$. $J_{\{0,1,\ldots,m\}}(n,2m+1)$ is not determined by its spectrum if $m \geq 2$ and $n \geq 4m+2$.*

Using GM-switching, we proved the following new result.

Theorem 10. *$K_q(n,k)$ is not determined by its spectrum if $q = 2$.*

4 Computational results

The following tables give a more structured overview of which cospectral results are already known for generalized Johnson and Grassmann graphs. The coloured boxes agree with the above theorems by the legend:

Theorem 4 | Theorem 5 | Theorem 6 | Theorem 7 | Theorem 8 | Theorem 9 | Theorem 10 | trivial (the disjoint union of copies of P_2 is DS, see [4])

For each graph of which the cospectrality is still unknown, we write down a number x. We searched exhaustively for possible GM-switching sets in each graph and found out that there do not exist GM-switching sets of size $\leq x$ with the property of producing a cospectral mate for the graph. Numbers between brackets indicate that we *did* find a switching set of that size, but the resulting cospectral graph was isomorphic to it.

Similar exhaustive searches have already been done in [3], but we extend it in two ways. First, our code is more efficient, and therefore able to produce higher numbers (and an extra find). Secondly, we add computational results for the generalized Grassmann graphs as well.

$J_S(n,2)$		S
		$\{0\}$
n	4	DS
	5	DS
	6	DS
	7	DS
	8	NDS
	9	DS

Table 1: Small generalized Johnson graphs with $k = 2$.

$J_S(n,3)$		S		
		$\{0\}$	$\{1\}$	$\{2\}$
n	6	DS	NDS	NDS
	7	DS	NDS	NDS
	8	NDS	NDS	NDS
	9	12	NDS	NDS
	10	12	NDS	NDS
	11	10	NDS	NDS

Table 2: Small generalized Johnson graphs with $k = 3$.

$J_S(n,4)$		S						
		$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$
n	8	DS	12	NDS	NDS	12	NDS	12
	9	DS	10	10	NDS	NDS	NDS	10
	10	8	8	8	NDS	8	NDS	8
	11	NDS	8	8	NDS	8	NDS	8
	12	6	6	6	NDS	6	NDS	6
	13	6	6	6	NDS	6	NDS	6

Table 3: Small generalized Johnson graphs with $k = 4$.

$J_S(n,5)$		S							
		$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$
n	10	DS	10	10	10	NDS	10	10	10
	11	DS	8	8	8	NDS	8	8	8
	12	6	6	6	6	NDS	NDS	6	6
	13	6	6	6	6	NDS	6	6	6
	14	NDS	4	4	4	NDS	4	4	4
	15	4	4	4	4	NDS	4	4	4

$J_S(n,5)$		S						
		$\{0,4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$
n	10	10	10 (4,8,10)	NDS	10 (4)	10 (4)	NDS	NDS
	11	8	8	NDS	8	6	6	NDS
	12	6	6	NDS	6	6	6	NDS
	13	6	4	NDS	4	4	4	NDS
	14	4	4	NDS	4	4	4	NDS
	15	4	4	NDS	4	4	4	NDS

Table 4: Small generalized Johnson graphs with $k = 5$.

$J_{q,S}(n,2)$		$q=2$	$q=3$	$q=4$
		S	S	S
		$\{0\}$	$\{0\}$	$\{0\}$
n	4	NDS	4	4
	5	NDS	NDS	NDS
	6	NDS	4 (4)	0
	7	NDS	0	0
	8	NDS	0	0
	9	NDS	0	0

Table 5: Small generalized Grassmann graphs with $k=2$ and $q=2,3,4$.

$J_{q,S}(n,3)$		$q=2$			$q=3$			$q=4$		
		S			S			S		
		$\{0\}$	$\{1\}$	$\{2\}$	$\{0\}$	$\{1\}$	$\{2\}$	$\{0\}$	$\{1\}$	$\{2\}$
n	6	NDS	0	NDS	0	0	NDS	0	0	NDS
	7	NDS	0	NDS	0	0	NDS	0	0	NDS
	8	NDS	0	NDS	0	0	NDS	0	0	NDS
	9	NDS	0	NDS	0	0	NDS	0	0	NDS
	10	NDS	0	NDS	0	0	NDS	0	0	NDS
	11	NDS	0	NDS	0	0	NDS	0	0	NDS

Table 6: Small generalized Grassmann graphs with $k=3$ and $q=2,3,4$.

$J_{q,S}(n,4)$		$q=2$							$q=3$						
		S							S						
		$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$
n	8	NDS	0	0	NDS	0	0	0	0	0	0	NDS	0	0	0
	9	NDS	0	0	NDS	0	0	0	0	0	0	NDS	0	0	0
	10	NDS	0	0	NDS	0	0	0	0	0	0	NDS	0	0	0
	11	NDS	0	0	NDS	0	0	0	0	0	0	NDS	0	0	0
	12	NDS	0	0	NDS	0	0	0	0	0	0	NDS	0	0	0
	13	NDS	0	0	NDS	0	0	0	0	0	0	NDS	0	0	0

$J_{q,S}(n,4)$		$q=4$						
		S						
		$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$
n	8	0	0	0	NDS	0	0	0
	9	0	0	0	NDS	0	0	0
	10	0	0	0	NDS	0	0	0
	11	0	0	0	NDS	0	0	0
	12	0	0	0	NDS	0	0	0
	13	0	0	0	NDS	0	0	0

Table 7: Small generalized Grassmann graphs with $k=4$ and $q=2,3,4$.

While searching for switching sets, we found a new graph that has a cospectral mate. This is the graph $J_{\{2,4\}}(10, 5)$. Its switching sets have size 10.

5 Conclusion

GM-switching provides a useful technique for discovering cospectral mates. It has been proven successful in the past, and it still is. WQH-switching is a recent new method that could be effective as well.

There are still many gaps in the tables. One way to deal with this, is to make the code run faster (for example, using multiple GPU's or ...). Another one is to try different switching techniques.

References

[1] A. Abiad and W. H. Haemers. "Cospectral graphs and regular orthogonal matrices of level 2". In: *Electron. J. Combin.* 19.3 (2012), Paper 13, 16. DOI: 10.37236/2383.

[2] C. Li-ch'ien. "The uniqueness and nonuniqueness of the triangular association schemes". In: *Sci. Record (N.S.)* 3 (1959), pp. 604–613.

[3] S. M. Cioabă, W. H. Haemers, T. Johnston, and M. McGinnis. "Cospectral mates for the union of some classes in the Johnson association scheme". In: *Linear Algebra Appl.* 539 (2018), pp. 219–228. DOI: 10.1016/j.laa.2017.11.011.

[4] E. R. van Dam and W. H. Haemers. "Which graphs are determined by their spectrum?" In: vol. 373. Special issue on the Combinatorial Matrix Theory Conference (Pohang, 2002). 2003, pp. 241–272. DOI: 10.1016/S0024-3795(03)00483-X.

[5] E. R. van Dam, W. H. Haemers, J. H. Koolen, and E. Spence. "Characterizing distance-regularity of graphs by the spectrum". In: *J. Combin. Theory Ser. A* 113.8 (2006), pp. 1805–1820. DOI: 10.1016/j.jcta.2006.03.008.

[6] E. R. van Dam and J. H. Koolen. "A new family of distance-regular graphs with unbounded diameter". In: *Invent. Math.* 162.1 (2005), pp. 189–193. DOI: 10.1007/s00222-005-0442-3.

[7] S. Dutta and B. Adhikari. "Construction of cospectral graphs". In: *J. Algebraic Combin.* 52.2 (2020), pp. 215–235. DOI: 10.1007/s10801-019-00900-y.

[8] C. D. Godsil and B. D. McKay. "Constructing cospectral graphs". In: *Aequationes Math.* 25.2-3 (1982), pp. 257–268. DOI: 10.1007/BF02189621.

[9] W. H. Haemers and F. Ramezani. "Graphs cospectral with Kneser graphs". In: *Combinatorics and graphs*. Vol. 531. Contemp. Math. Amer. Math. Soc., Providence, RI, 2010, pp. 159–164. DOI: 10.1090/conm/531/10465.

[10] A. J. Hoffman. "On the uniqueness of the triangular association scheme". In: *Ann. Math. Statist.* 31 (1960), pp. 492–497. DOI: 10.1214/aoms/1177705914.

[11] T. Huang and C.-R. Liu. "Spectral characterization of some generalized odd graphs". In: *Graphs Combin.* 15.2 (1999), pp. 195–209. DOI: 10.1007/s003730050040.

[12] J. van Lint and J. Seidel. "Equilateral point sets in elliptic geometry". In: *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen: Series A: Mathematical Sciences* 69.3 (1966), pp. 335–348.

[13] W. Wang, L. Qiu, and Y. Hu. "Cospectral graphs, GM-switching and regular rational orthogonal matrices of level p". In: *Linear Algebra Appl.* 563 (2019), pp. 154–177. DOI: 10.1016/j.laa.2018.10.027.

Advances on the 3-color off-diagonal generalized Schur numbers $S(3; k_1, k_2, k_3)$

T. Ahmed[1*], L. Boza[2†], M. P. Revuelta[3†], M. I. Sanz[4†]

[*]Department of Computer Science and Software Engineering Concordia University, Montréal, Canada
[†]Departamento de Matemática Aplicada I, Universidad de Sevilla, Spain

Abstract

For integers r and k_i, with $r \geq 2$ and $k_i \geq 2$ for $i = 1, 2, \ldots, r$, the *r-color off-diagonal generalized Schur number* $S(r; k_1, k_2, \ldots, k_r)$ is defined as the least integer M such that any r-coloring of the integer interval $[1, M]$ must admit a j-colored solution to equation $E_{k_j} : x_1 + x_2 + \ldots + x_{k_j} = x_{k_j+1}$ for some j with $1 \leq j \leq r$.

In this work, we will show our advances on the 3-color off-diagonal generalized Schur numbers $S(3; k_1, k_2, k_3)$.

1 Introduction

For integers $a \leq b$, we shall denote $[a, b]$ the *integer interval* consisting of all $t \in \mathbb{N}_+ = \{1, 2, \ldots\}$ such that $a \leq t \leq b$. A function

$$\Delta : [1, N] \longrightarrow \{d_1, \ldots, d_r\},$$

where $d_1, \ldots, d_r \in \mathbb{N}_+$ represent different colors, is an r-coloring of the interval $[1, N]$.

Given an r-coloring Δ and the equation $E_k : x_1 + \cdots + x_k = x_{k+1}$ in $k+1$ variables, we say that a solution $x_1, \ldots, x_k, x_{k+1}$ to the equation E_k is monochromatic if and only if $\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_{k+1})$.

1.1 Schur numbers

A set A of integers is called *2-sum-free* if it contains no elements $x_1, x_2, x_3 \in A$ satisfying $x_1 + x_2 = x_3$ where x_1, x_2 need not be distinct.

Given a positive integer $r \geq 2$, Schur [15] proved in 1916 that there exists a greatest positive integer N such that the integer interval $[1, N-1]$ can be partitioned into r *2-sum-free* sets. Such numbers are denoted by $S(r)$ in the literature. For notational convenience, we denote $S(r)$ equivalently by $S(r; 2, 2, \ldots, 2)$ where the number of 2's is equal to r and the 2's are due to the 2-sums in the equation $E_2 : x_1 + x_2 = x_3$. The current knowledge on these numbers for $1 \leq r \leq 7$ is given in Table 1.

r	1	2	3	4	5	6	7
$S(r; 2, 2, \ldots, 2)$	2	5	14	45	161	≥ 537	≥ 1681

Table 1: The first few Schur numbers $S(r; 2, 2, \ldots, 2)$.

[1]Email: tanbir@gmail.com
[2]Email: boza@us.es
[3]Email: pastora@us.es
[4]Email: isanz@us.es

The exact value of $S(4;2,2,2,2)$ was obtained by Baumert and Golomb ([5], [9]) and recently $S(5;2,2,2,2,2)$ has been obtained by Heule [11]. Finally, the lower bounds on $S(6;2,2,2,2,2,2)$ and $S(7;2,2,2,2,2,2,2)$ were obtained by Fredricksen and Sweet [8] by considering symmetric 2-sum-free partitions.

1.2 Generalized Schur numbers

In 1933, Rado [12] gave the following generalization: given two positive integers r and k such that $r \geq 2$, $k \geq 2$, there exists a greatest positive integer, $M = S(r;k,k,\ldots,k)$, where the number of k's equals r, and the k's are due to the k sums in the equation $E_k : x_1 + x_2 + \ldots + x_k = x_{k+1}$, such that the integer interval $[1, M-1]$ can be partitioned into r sets which are *k-sum-free*.

In 1966, Znám [16] established a lower bound on the numbers $S(r;k,k,\ldots,k)$:

$$S(r;k,k,\ldots,k) \geq \frac{k-1}{k}((k+1)^n - 1) + 1.$$

In 1982, Beutelspacher and Brestovansky [6] proved that $S(2;k,k) = k^2 + k - 1$ for all $k \geq 2$.

In 2010 [14], the last author obtained the exact value of $S(3;3,3,3) = 43$. Independently, Ahmed and Schaal [4] in 2016 gave the values of $S(3;k,k,k)$ for $k = 3,4,5$. In 2019, Boza et al. [7] determined the exact formula of $S(3;k,k,k) = k^3 + 2k^2 - 2$ for all $k \geq 3$, finding an upper bound that coincides with the lower bound given by Znám [16].

The following extension of the *generalized Schur numbers* was defined by Robertson and Schaal in [13]. Let $r \geq 2$ and $k_i \geq 2$ for $i = 1,\ldots,r$, the *r-color off-diagonal generalized Schur number*, denoted by $S(r;k_1,k_2,\ldots,k_r)$, is defined as the least integer M such that any r-coloring of the integer interval $[1,M]$ must admit a j-colored solution to equation $E_{k_j} : x_1 + x_2 + \ldots + x_{k_j} = x_{k_j+1}$ for some j with $1 \leq j \leq r$. These numbers are given their name because of their similarity to the classical off-diagonal Ramsey numbers. Robertson and Schaal [13] determined all values of the 2-color off-diagonal generalized Schur numbers, $S(2;k_1,k_2)$ with $k_i \geq 2$ for $i = 1,2$.

2 Main results

We have determined the exact values of $S(3;2,2,k)$, with $k \geq 2$ in the following result:

Theorem 1. *[3] For all $k \geq 2$, we have*

$$S(3;2,2,k) = \begin{cases} 9k-4 & \text{if} \quad k \notin 1+5\mathbb{N} \\ 9k-5 & \text{if} \quad k \in 1+5\mathbb{N} \end{cases}$$

In addition, we have found a general lower bound of the 3-color off-diagonal generalized Schur numbers $S(3;2,k_1,k_2)$ with $k_2 \geq k_1 \geq 3$ in the following result:

Theorem 2. *[1] A lower bound of $S(3;2,k_1,k_2)$, with $k_2 \geq k_1 \geq 3$, is:*

- *When k_2 is odd:*
  1. *If $3 \leq k_1 \leq \left\lfloor \frac{k_2+7}{5} \right\rfloor$ then $S(3;2,k_1,k_2) \geq 3k_1k_2 + k_2 - 2k_1 + 2$.*
  2. *If either $\left\lceil \frac{k_2+8}{5} \right\rceil \leq k_1 \leq k_2 - 2$ or $k_1 = k_2$ then $S(3;2,k_1,k_2) \geq 3k_1k_2 + 3k_1 - 5$.*
  3. *If $k_2 = 5$ and $k_1 = 4$ then $S(3;2,k_1,k_2) \geq 70$.*
  4. *If $6 \leq k_1 = k_2 - 1$ then $S(3;2,k_1,k_2) \geq 3k_2^2 - 3k_2 - 3$.*
- *When k_2 is even:*

1. *If* $3 \leq k_1 \leq \left\lfloor \frac{k_2}{4} \right\rfloor + 1$ *then* $S(3;2,k_1,k_2) \geq 3k_1k_2 + k_2 - k_1 + 1$.
2. *If* k_1 *is odd and* $\left\lfloor \frac{k_2}{4} \right\rfloor + 2 \leq k_1 \leq k_2 - 3$ *then* $S(3;2,k_1,k_2) \geq 3k_1k_2 + 3k_1 - 4$.
3. *If* k_1 *is even and* $\left\lfloor \frac{k_2}{4} \right\rfloor + 2 \leq k_1 \leq k_2 - 4$ *then* $S(3;2,k_1,k_2) \geq 3k_1k_2 + 3k_1 - 5$.
4. *If* $k_1 = k_2 - 2 \geq 4$ *then* $S(3;2,k_1,k_2) \geq 3k_2^2 - 3k_2 - 10$.
5. *If* $k_1 = k_2 - 1 \geq 3$ *then* $S(3;2,k_1,k_2) \geq 3k_2^2 - 1$.
6. *If* $k_1 = k_2 \geq 4$ *then* $S(3;2,k_1,k_2) \geq 3k_2^2 + 3k_2 - 2$.

The values of 3-color off-diagonal generalized Schur numbers $S(3;2,3,k)$ with $3 \leq k \leq 6$ were determined by the first author and Schaal in [4].

Finally, we have determined the exact values of $S(3;2,3,k)$ with $k \geq 7$.

Theorem 3. *[2] Let* $k \geq 8$ *even, we have* $S(3;2,3,k) = 10k - 2$.

Theorem 4. *[2]* $S(3;2,3,7) = 67$ *and if* $k \geq 9$ *odd then* $S(3;2,3,k) = 10k - 4$.

3 An idea of the methods involved in reaching these results

3.1 Sketch of the Theorem 1 proof

Firstly, we will show the lower bound of $S(3;2,2,k)$ with $k \geq 2$.
We shall denote $I_p[n,m] = [n,m] \cap (p + 5\mathbb{N})$ for $p \in \{0,1,2,3,4\}$.

Lemma 5. *If* $k \in 5\mathbb{N}$ *then* $S(3;2,2,k) \geq 9k - 4$.

Proof. Let $A_1 = I_2[2,3k-3] \cup I_3[3,9k-7] \cup I_4[6k-1,9k-6]$ $A_2 = I_1[1,4k-4] \cup I_4[4,k-1] \cup I_0[5k,9k-5] \cup I_2[8k-3,9k-8]$ $A_3 = I_0[5,5k-5] \cup I_4[k+4,6k-6] \cup I_2[3k+2,8k-8] \cup I_1[4k+1,9k-9]$
We consider a 3-coloring of $[1,9k-5]$:

$$\Delta : [1,9k-5] \longrightarrow \{d_1,d_2,d_3\},$$

where d_1,d_2,d_3 represent 3 different colors. Let $\Delta(A_i) = d_i$ for $i = 1,2,3$ such that

$$[1,9k-5] = A_1 \sqcup A_2 \sqcup A_3.$$

We will prove that A_1 and A_2 have no monochromatic solutions to the equation E_2 of colors d_1 or d_2, and A_3 has no monochromatic solutions to the equation E_k of color d_3.

□

Similarly, we will show the following results:

Lemma 6. *If* $k \in 1 + 5\mathbb{N}$ *then* $S(3;2,2,k) \geq 9k - 5$.

Lemma 7. *If* $k \in 2 + 5\mathbb{N}$ *then* $S(3;2,2,k) \geq 9k - 4$.

Lemma 8. *If* $k \in 3 + 5\mathbb{N}$ *then* $S(3;2,2,k) \geq 9k - 4$.

Finally, we will make use of computer assisted to proof the upper bound of $S(3;2,2,k)$ with $k \geq 2$.

3.2 Sketch of the Theorem 2 proof

We will show a lower bound of $S(3;2,k_1,k_2)$ with $3 \leq k_1 \leq k_2$. So far only the values $k_2 \leq 6$ were determined by Ahmed and Schaal [4].
We shall denote $J_p[n,m] = [n,m] \cap (p+2\mathbb{N})$ with $p \in \{0,1\}$.

Lemma 9. *If $k_2 \in 1+2\mathbb{N}$ then $S(3;2,k_1,k_2) \geq 3k_1k_2 + k_2 - 2k_1 + 2$.*

Proof.

- If $k_1 \in 1+2\mathbb{N}$, let

$$\begin{aligned} A_1 &= J_1[1,k_2-2] \cup J_0[2k_2,4k_2-1] \cup J_1[3k_1k_2-2k_1, 3k_1k_2+k_2-2k_1+1]. \\ A_2 &= J_1[3k_2-2,4k_2-1] \cup [4k_2, 3k_1k_2-2k_1-1] \cup J_0[3k_1k_2-2k_1, 3k_1k_2+k_2-2k_1+1]. \\ A_3 &= J_0[1,k_2-1] \cup [k_2,2k_2-1] \cup J_1[2k_2,3k_2-3]. \end{aligned}$$

- If $k_1 \in 2\mathbb{N}$, let

$$\begin{aligned} A_1 &= J_1[1,k_2-2] \cup J_0[2k_2,4k_2-1] \cup J_0[3k_1k_2-2k_1, 3k_1k_2+k_2-2k_1+1]. \\ A_2 &= J_1[3k_2-2,4k_2-1] \cup [4k_2, 3k_1k_2-2k_1-1] \cup J_1[3k_1k_2-2k_1, 3k_1k_2+k_2-2k_1+1]. \\ A_3 &= J_0[1,k_2-1] \cup [k_2,2k_2-1] \cup J_1[2k_2,3k_2-3]. \end{aligned}$$

We consider a 3-coloring of $[1, 3k_1k_2+k_2-2k_1+1]$:

$$\Delta : [1, 3k_1k_2+k_2-2k_1+1] \longrightarrow \{d_1,d_2,d_3\},$$

where d_1,d_2,d_3 represent 3 different colors and $\Delta(A_i) = d_i$ for $i=1,2,3$ such that

$$[1, 3k_1k_2+k_2-2k_1+1] = A_1 \sqcup A_2 \sqcup A_3.$$

We will prove that A_1 has no monochromatic solutions to the equation E_2 of color d_1, A_2 has no monochromatic solutions to the equation E_{k_1} of color d_2 and A_3 has no monochromatic solutions to the equation E_{k_2} of color d_3.

□

Similarly, we will show the following results:

Lemma 10. *If $k_2 \in 2\mathbb{N}$ then $S(3;2,k_1,k_2) \geq 3k_1k_2 + k_2 - k_1 + 1$.*

Lemma 11. *If $k_1 \in 2\mathbb{N}$ or $k_2 \in 1+2\mathbb{N}$ then $S(3;2,k_1,k_2) \geq 3k_1k_2 + 3k_1 - 5$.*

Lemma 12. *If $k_1 \in 1+2\mathbb{N}$ and $k_2 \in 2\mathbb{N}$ then $S(3;2,k_1,k_2) \geq 3k_1k_2 + 3k_1 - 4$.*

3.3 Sketch of the Theorem 3 proof and Theorem 4 proof

We will make use of computer assisted to proof the upper bound of of $S(3;2,3,k)$ with $k \geq 7$.

References

[1] T. Ahmed, L. Boza, M. Revuelta, and M. Sanz. *A general lower bound of the 3-color off-diagonal generalized Schur numbers $S(3;2,k_1,k_2)$*. Preprint. 2022.

[2] T. Ahmed, L. Boza, M. Revuelta, and M. Sanz. *Exact values of the 3-color off-diagonal generalized Schur numbers $S(3;2,3,k)$*. Preprint. 2022.

[3] T. Ahmed, L. Boza, M. Revuelta, and M. Sanz. *On the 3-color off-diagonal generalized Schur numbers* $S(3; 2, 2, k)$. Preprint. 2022.

[4] T. Ahmed and D. J. Schaal. "On generalized Schur numbers". In: *Exp. Math.* 25.2 (2016), pp. 213–218. DOI: 10.1080/10586458.2015.1070776.

[5] L. Baumert. *Sum-free sets*. J.P.L. Research Summary. 1961.

[6] A. Beutelspacher and W. Brestovansky. "Generalized Schur numbers". In: *Combinatorial theory (Schloss Rauischholzhausen, 1982)*. Vol. 969. Lecture Notes in Math. Springer, Berlin-New York, 1982, pp. 30–38.

[7] L. Boza, J. M. Marín, M. P. Revuelta, and M. I. Sanz. "3-color Schur numbers". In: *Discrete Appl. Math.* 263 (2019), pp. 59–68. DOI: 10.1016/j.dam.2018.06.030.

[8] H. Fredricksen and M. M. Sweet. "Symmetric sum-free partitions and lower bounds for Schur numbers". In: *Electron. J. Combin.* 7 (2000), Research Paper 32, 9.

[9] S. W. Golomb and L. D. Baumert. "Backtrack programming". In: *J. Assoc. Comput. Mach.* 12 (1965), pp. 516–524. DOI: 10.1145/321296.321300.

[10] M. Heule. *March RW. SAT Competition 2011*. http://www.st.ewi.tudelft.nl/sat/. 2011.

[11] M. J. H. Heule. "Schur Number Five". In: *The Thirty-Second AAAI Conference on Artificial Intelligence (AAAI-18)*. New Orleans, Louisiana, USA: AAAI Press, 2018.

[12] R. Rado. "Studien zur Kombinatorik". In: *Math. Z.* 36.1 (1933), pp. 424–470. DOI: 10.1007/BF01188632.

[13] A. Robertson and D. Schaal. "Off-diagonal generalized Schur numbers". In: *Adv. in Appl. Math.* 26.3 (2001), pp. 252–257. DOI: 10.1006/aama.2000.0718.

[14] M. I. Sanz. "Números de Schur y Rado". PhD thesis. Universidad de Sevilla, 2010.

[15] I. Schur. "Über die Kongruenz $x^m + y^m \equiv z^m (\mod p)$". In: *Jber. Deutsch. Math.- Verein.* 25 (1916), pp. 114–117.

[16] Š. Znám. "Generalisation of a number-theoretical result". In: *Mat.-Fyz. Časopis. Sloven. Akad. Vied.* 16 (4 1966), pp. 357–361.

On bipartite biregular Moore graphs

Gabriela Araujo-Pardo[1*], Cristina Dalfó[2†], Miguel Àngel Fiol[3‡], Nacho López[4§]

[*]Instituto de Matematicas, Universidad Nacional Autónoma de México, México
[†]Departament de Matemàtica, Universitat de Lleida, Igualada (Barcelona), Catalonia.
[‡]Departament de Matemàtiques, Universitat Politècnica de Catalunya, Barcelona Graduate School of Mathematics. Institut de Matemàtiques de la UPC-BarcelonaTech (IMTech), Barcelona, Catalonia.
[§]Departament de Matemàtica, Universitat de Lleida, Lleida, Spain.

The full version of this work can be found in [1].

Abstract

The *degree/diameter problem* for graphs consists in finding the largest order of a graph with prescribed degree and diameter, the *Moore bound* is the upper bound of this order for any pair of fixed values (diameter and degree). If there exists a graph whose order coincides with this bound we call it a *Moore graph*.
A bipartite graph $G = (V, E)$ with $V = V_1 \cup V_2$ is biregular if all the vertices of a stable set V_i have the same degree r_i for $i = 1, 2$. In this work, we study the *diameter/degree problem* in this context, introduced in 1983 by Yebra, Fiol, and Fàbrega.
The authors of that paper give a Moore bound for bipartite biregular graphs, called the *Moore-like bound*. In this work, we proved that for some cases of odd diameter it is impossible to attain this bound and we give a new bound for these specific cases, we said that we improve the Moore-like bound. We also propose some constructions of large bipartite biregular graphs, some of them attaining our new Moore-like bound.

1 Introduction

The *degree/diameter problem* for graphs consists in finding the largest order of a graph with prescribed degree and diameter, the *Moore bound* is the upper bound of this order for any pair of fixed values (diameter and degree). If there exists a graph whose order coincides with this bound we call it a *Moore graph*.
There exist a lot of work related to this topic (see the survey by Miller and Širáň [7]). In this work, we study the problem, proposed by Yebra, Fiol, and Fàbrega [8] in 1983, consisting in finding biregular bipartite Moore graphs.

A bipartite graph $G = (V, E)$ with $V = V_1 \cup V_2$ is biregular if, for $i = 1, 2$, all the vertices of a stable set V_i have the same degree. We denote $[r, s; d]$-bigraph a bipartite biregular graph of degrees r and s, for $r \geq s$ and diameter d; and by $[r, s; d]$-bimoore graph the bipartite biregular graph of diameter d that attains the Moore-like bound given in [8], and denoted as $M(r, s; d)$ (1). In this work, we improve

[1]Email: garaujo@im.unam.mx, Research of G. A. supported by CONACyT-México under Project 282280 and PAPIIT-México Projects IN107218, IN106318, IN101821 and PASPA-DGAPA and CONACyT Sabbatical Year 2020

[2]Email: cristina.dalfo@udl.cat, Research of R. J. supported by European Union's Horizon 2020 research and innovation program under the Marie Skłodowska-Curie grant agreement No 734922, AGAUR from the Catalan Government under project 2017SGR1087 and by MICINN from the Spanish Government under project PGC2018-095471-B-I00 and MICINN from the Spanish Government under project MTM2017-83271-R.

[3]miguel.angel.fiol@upc.edu, Research of A. R. supported by AGAUR from the Catalan Government under project 2017SGR1087 and by MICINN from the Spanish Government under project PGC2018-095471-B-I00.

[4]nacho.lopez@udl.cat, Research of A. R. supported by MICINN from the Spanish Government under project MTM2017-83271-R.

that bound, and consequently, the graphs that attain this improved Moore-like bound and also called bimoore graphs.
Notice that constructing these graphs is equivalent to finding block designs, where one partite set corresponds to the points of the block design, and the other set corresponds to the blocks of the design. Moreover, each point is in a fixed number s of blocks, and the size of each block is equal to r. The incidence graph of this block design is an $[r, s; d]$-biregular bipartite graph of diameter d. The degree/diameter problem is strongly related to the *degree/girth problem* (also known as *the cage problem*) that consists in finding the smallest order of a graph with prescribed degree and girth (see the survey by Exoo and Jajcay [4]). Notice that when the girth of the graph is odd $g = 2d + 1$, the lower bound of this value coincides with the Moore bound for graphs (the largest order of a regular graph with a given diameter d). Furthermore, when we restrict our problem to bipartite graphs, for graphs of even girth $g = 2d$, the lower bound for the cage problem coincides with the Moore bound for bipartite graphs of diameter d.

In the bipartite biregular problem, we have the same situation. In 2019, Filipovski, Ramos-Rivera and Jajcay [6] introduced the concept of bipartite biregular Moore cages and presented lower bounds on the orders of bipartite biregular $(m, n; g)$-graphs. The bounds when $g = 2d$ and d even also coincide with the bounds given by Yebra, Fiol, and Fàbrega in [8]. It is important to note that when we have an odd diameter d and girth $g = 2d$ the lower bound for cages is different than the upper bound for Moore graphs and the techniques to calculate them are different.

Before starting and enlisting our results on this extended abstract, we would like to emphasize that we approach the problem using different techniques: results on finite geometries and generalized quadrangles, results on spectral graph theory, and we obtain values thanks to computational calculations. The values of the Moore-like bound for a biregular bipartite $[r, s; d]$-graph,, given in [8], are the following:

$$M(r,s;d) = \begin{cases} (r+s)\frac{[(r-1)(s-1)]^m-1}{(r-1)(s-1)-1}, & \text{if } d = 2m \text{ is even;} \\ \left\lfloor \frac{1+s(r-1)\frac{[(r-1)(s-1)]^m-1}{(r-1)(s-1)-1}}{\rho} \right\rfloor (\rho+\sigma), & \text{if } d = 2m+1 \text{ is odd.} \end{cases} \tag{1}$$

Where $\rho = \frac{r}{\gcd\{r,s\}}$ and $\sigma = \frac{s}{\gcd\{r,s\}}$.

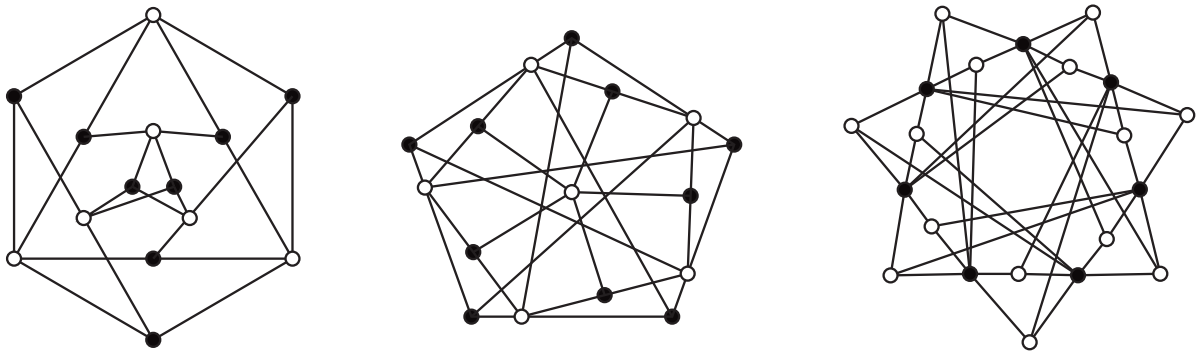
Recall that if $G = (V, E)$ is an r-regular graph of diameter d, then its *defect* is $\delta = \delta(G) = M(r; d) - |V|$, where $M(r; d)$ stands for the corresponding Moore bound. Thus, in this work, the defect of a $[r, s; d]$-bigraph $G = (V_1 \cup V_2, E)$ is defined as $\delta = M(r, s; d) - |V_1 \cup V_2|$.

2 Our contributions

In this section, we give our main results, four tables with the best current values for every case. The attainable known values are in boldface, the asterisks correspond to the graphs obtained according to Proposition 6, the diamonds correspond to unique graphs, the values between parenthesis correspond to the old (unattainable) bound (1) and the symbol '•' indicates the orders of the graphs according to Theorem 3.

2.1 The case of diameter 3

In Figure 1 we show two bimoore graphs with diameter three attaining the Moore bound (1), which were given in [8] for $s = 3$ and $r = \{4, 5\}$, and the unique $[6, 3; 3]$-bimoore graph with 21 vertices. Notice that the Moore bound given in [8] $M(6, 3; 3) = 24$. In the in the following proposition we improve this upper bound for some specific values of r and s.



(a) (b) (c)

Figure 1: (*a*) The only [4,3;3]-bimoore graph on 14 vertices; (*b*) One of the two [5,3;3]-bimoore graph on 16 vertices; (*c*) The only [6,3;3]-bimoore graph on 21 vertices.

$r \setminus s$	2	3	4	5	6	7	8	9	10	11
2	**6**									
3	**5**	**14**								
4	**9**	**14**$^{*\diamond}$	**26**							
5	**7**	**16***	27	**42**						
6	**12**	(24) **21**$^{\diamond}$	(35) 30	44	**62**					
7	**9**	**20***	33	48	65	86				
8	**15**	**22***	42	52	70	90	**114**			
9	**11**	32	39	56	80	96	119	**146**		
10	**18**	**26***	49	(69) 66	(88) 80	102	126	152	**182**	
11	**13**	**28***	45	64	85	108	133	160	189	222

Table 1: Best Moore bounds for diameter $d = 3$.

Proposition 1. *If* $\rho = \frac{r}{\gcd\{r,s\}}$ *divides* $s-1$*, then there is no* $[r,s;3]$*-graph with order attaining the Moore-like bound in* (1)*. Instead, the new improved Moore bound is*

$$M^*(r,s;3) = (1 + s(r-1) - \rho)\left(1 + \frac{\sigma}{\rho}\right), \tag{2}$$

Where $\rho = \frac{r}{\gcd\{r,s\}}$ *and* $\sigma = \frac{s}{\gcd\{r,s\}}$.

As a consequence of this result, there is no exists a $[6,3;3]$-bipartite biregular graph with order 24, the new bound is 21 and the graph depicted in Figure 1 (*c*) is a $[6,3;3]$-bimoore graph. In the sequel, we will show how to construct it.

2.2 Relation between bipartite biregular Moore graphs and generalized polygons

The connection between Moore graphs and generalized polygons was extensively studied (see, for instance, Bamberg, Bishnoi, and Royle [3]). In 1964, Feit and Higman proved that finite generalized n-gons exist only for $n = \{3,4,6,8\}$. When $n = 3$, we have the projective planes; when $n = 4$, we

$r \setminus s$	2	3	4	5	6	7	8	9	10
2	**8**$^{\bullet}$								
3	**15**$^{*\diamond}$	**30**							
4	**24**$^{*\diamond}$	49	**80**						
5	**35***	**72**$^{\bullet}$	117	**170**					
6	**48***	99	160	231	**312**				
7	**63***	130	209	300	403	(518) 516			
8	**80***	165	264	377	504	645	**800**		
9	**99***	204	325	462	615	784	969	**1170**	
10	**120***	247	**292**$^{\bullet}$	555	736	935	1152	1387	**1640**

Table 2: Moore bounds for diameter $d = 4$.

$r \setminus s$	2	3	4	5	6	7	8	9	10
2	10								
3	**15**$^{\diamond}$	62							
4	36	(112) 105	242						
5	56	168	369	682					
6	80	(249) 246	(535) 530	957	1562				
7	108	330	715	(1284) 1272	2067	3110			
8	140	429	924	(1651) 1638	2646	3945	5602		
9	176	544	(1157) 1144	2044	3280	4880	6885	9362	
10	216	663	1407	(2499) 2496	(3976) 3968	5882	8289	11229	14762

Table 3: Best Moore bounds for diameter $d = 5$.

have the generalized quadrangles, which are known to exist for parameter pairs $(q, q), (q, q^2), (q^2, q)$, $(q^2, q^3), (q^3, q^2), (q-1, q+1), (q+1, q-1)$; when $n = 6$, we have the generalized hexagons with parameters $(q, q), (q, q^3), (q^3, q)$, in both cases for q prime power; and, finally, for $n = 8$, we have the generalized octagons, which are only known to exist for the pairs $(q, q^2), (q^2, q)$, where q is an odd power of 2. In [2] the authors proved that:

Theorem 2. *[2] Whenever a generalized quadrangle, hexagon, or octagon $\mathcal{G}$ of order (s, t) exists, its point-line incidence graph is an $(s+1, t+1; 8)$-, $(s+1, t+1; 12)$- or $(s+1, t+1; 16)$-cage, respectively. Hence, there exist infinite families of bipartite biregular $(n+1, n^2+1; 8)$-, $(n^2+1, n^3+1; 8)$-, $(n, n+2; 8)$-, $(n+1, n^3+1; 12)$- and $(n+1, n^2+1; 16)$-cages.*

Taking into account the relationship between bipartite biregular cages of girth $g = 2d$ and bipartite biregular Moore graphs for diameter even d given in the introduction, we conclude the following result.

Theorem 3. *There exists infinite families of bipartite biregular $[r^2+1, r+1; 4]$-, $[r^3+1, r^2+1; 4]$-, $[r+2, r; 4]$-, $[r^3+1, r+1; 6]$- and $[r^2+1, r+1; 8]$-bimoore graphs.*

In [1], we have two constructions that apply results about extremal graphs and generalized quadrangles to obtain bimoore graphs. One of them provides graphs with degrees r and 2 and diameter $d = 2m$ of order $M(r, 2; d) = \frac{r+2}{r-2}[(r-1)^m - 1]$ (see [1]), and the other is as follows:
Let G be a bipartite biregular graph with stable sets V_1 and V_2. Given $i \in \{1, 2\}$, the *semi-double(-V_i)*

$r \setminus s$	2	3	4	5	6	7	8	9	10
2	**12•**								
3	**35***	**126**							
4	**78***	301	**728**						
5	**147***	584	1431	**2730**					
6	**248***	999	2410	4631	**7812**				
7	387	1570	3773	7212	12103	(18662) 18660			
8	**570***	2321	5556	10569	17654	27105	**39216**		
9	**803***	3276	**7813•**	14798	24615	37648	54281	**74898**	
10	**1092***	4459	10598	19995	33136	50507	72594	99883	**132860**

Table 4: Moore bounds for diameter $d = 6$.

graph G^{2V_i} is obtained from G by doubling each vertex of V_i, so that each vertex $u \in V_i$ gives rise to another vertex u' with the same neighborhood as u, $G(u') = G(u)$. Thus, assuming, without loss of generality, that $i = 1$, the graph G^{2V_1} is bipartite with stable sets $V_1 \cup V_1'$ and V_2, and satisfies the following result.

Theorem 4. *Let $G = (V_1 \cup V_2, E)$ be a bipartite biregular graph on $n = n_1 + n_2 = |V_1| + |V_2|$ vertices, diameter $D(\geq 2)$, and spectrum* $\operatorname{sp} G$.

Then, its semi-double graph G^{2V_1}, on $N = 2n_1 + n_2$ vertices, has the same diameter D, and spectrum

$$\operatorname{sp} G^{2V_1} = \sqrt{2} \cdot \operatorname{sp} G \cup \{0^{n_1}\}. \tag{3}$$

As a consequence of Theorem 4 and the incidence graphs of the generalized polygons, the bipartite Moore graphs of order $M(r; d)$ for $r - 1$ and $d = \{3, 4, 6\}$, we introduce a family of $[r, 2r; d]$-graphs.

Theorem 5. *The following are $[2r, r; d]$-bipartite biregular graphs for $r \geq 3$, $r - 1$ a prime power, and diameter $d \in \{3, 4, 6\}$.*

(i) A $[2r, r; 3]$-bipartite biregular graph has order $n = 3r^2 - 3r + 3$ with defect $\delta = \frac{3}{2}(r - 1)$ for odd r, and $\delta = \frac{3r}{2} - 3$ for even r.

(ii) A $[2r, r; 4]$-bipartite biregular graph has order $n = 3r^3 - 6r^2 + 10r$ with defect $\delta = 3r^3 - 3r^2 - 2r$.

(iii) A $[2r, r; 6]$-bipartite biregular graph has order $n = 2r^5 - 8r^4 + 14r^3 - 12r^2 + 6r$ with defect $\delta = 10r^5 - 28r^4 + 31r^3 - 15r^2 - 3r$.

Notice that applying Theorem 5(i) to the Heawood graph (the incidence graph of the Fano plane), we obtain the $[6, 3; 3]$-bimoore graph of order 21 depicted in Figure 1(*c*).

In [1] we give a general construction for $s = 3$ and $d = 3$ that give us the following result.

Theorem 6. *For any integer $r \geq 6$ such that $3 \nmid r$, there exists a bipartite graph G on $2r + 6$ vertices (the Moore bound for the case of degrees $(r, 3)$ and diameter 3) with degrees $3, r, r \pm 1$ and diameter $d = 3$. Moreover, when $r \equiv 2 \mod 3$, there exists a Moore bipartite biregular graph with degrees $(r, 3)$ and diameter 3.*

Finally, it is worthy to say that we obtain some values with computational analysis. In particular for $s = 3$ and $d = 3$, we obtain results giving us the number of graphs for any r.

References

[1] G. Araujo-Pardo, C. Dalfó, M. Fiol, and N. López. "Bipartite biregular Moore graphs". In: *Discrete Mathematics* 344.11 (2021), p. 112582. DOI: https://doi.org/10.1016/j.disc.2021.112582.

[2] G. Araujo-Pardo, R. Jajcay, A. Ramos-Rivera, and T. Szőnyi. "On a relation between bipartite biregular cages, block designs and generalized polygons". In: *J. Comb. Des.* 30.7 (2022), pp. 479–496. DOI: https://doi.org/10.1002/jcd.21836. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/jcd.21836.

[3] J. Bamberg, A. Bishnoi, and G. F. Royle. "On regular induced subgraphs of generalized polygons". In: *J. Combin. Theory Ser. A* 158 (2018), pp. 254–275. DOI: 10.1016/j.jcta.2018.03.015.

[4] G. Exoo and R. Jajcay. "Dynamic cage survey". In: *Electron. J. Combin.* DS16.Dynamic Surveys (2008), p. 48.

[5] W. Feit and G. Higman. "The nonexistence of certain generalized polygons". In: *J. Algebra* 1 (1964), pp. 114–131. DOI: 10.1016/0021-8693(64)90028-6.

[6] S. Filipovski, A. R. Rivera, and R. Jajcay. "On biregular bipartite graphs of small excess". In: *Discrete Math.* 342.7 (2019), pp. 2066–2076. DOI: 10.1016/j.disc.2019.04.004.

[7] M. Miller and J. Širáň. "Moore graphs and beyond: a survey of the degree/diameter problem". In: *Electron. J. Combin.* DS14.Dynamic Surveys (2005), p. 61.

[8] J. L. A. Yebra, M. A. Fiol, and J. Fàbrega. "Semiregular bipartite Moore graphs". In: *Ars Combin.* 16.A (1983), pp. 131–139.

Computation of 2D discrete geometric moments through inclusion-exclusion

Lidija Čomić[1*], Paola Magillo[2†]

[*]University of Novi Sad, Faculty of Technical Sciences, Novi Sad, Serbia
[†]University of Genova, DIBRIS, Genova, Italy

The full version of this work can be found in [1] and will be published elsewhere.

Abstract

We propose a new formula for computing discrete geometric moments on 2D binary images, based on inclusion-exclusion. If applied to images characterized by a prevalence of horizontal and vertical lines, it reduces the number of pixels where calculations are to be performed.

1 Background Notions

We consider an object (a set of black pixels) in a raster of $N \times M$ pixels, with other pixels being white.

1.1 Discrete Geometric (Cartesian) Moments

The geometric moment of a digital object O is

$$m_{p,q}(O) = \sum_{(i,j)\in O} i^p j^q. \tag{1}$$

When the digital object is a rectangle R composed of pixels centered at points in $[1, J] \times [1, L] \cap \mathbb{N}^2$, with J, L integers,

$$m_{p,q}(R(J, L)) = \sum_{i=1}^{J} i^p \sum_{j=1}^{L} j^q = S_p(J) \cdot S_q(L). \tag{2}$$

where S_p and S_q denote the sum of exponentials, defined as $S_k(n) = \sum_{h=1}^{n} h^k$.

1.2 Discrete Green's Theorem

The discrete version of Green's theorem, in the formulation proposed by Tang [11], states that

$$\sum_{(i,j)\in O} f(i,j) = \sum_{(i,j)\in \mathcal{C}(O)} \left(F_x(i,j)D_Y(i,j) + f(i,j)C_Y(i,j)\right), \tag{3}$$

where $\mathcal{C}(O)$ is the set of contour pixels of O (i.e., the pixels of O that are edge-adjacent to at least one white pixel), $F_x(i,j) = \sum_{n=0}^{i} f(n,j)$, $D_Y(i,j) = 1$ or -1 if (i,j) is the first or the last pixel of a run (a maximal set of contiguous black pixels in one row) with length > 1, otherwise it is 0; $C_Y(i,j) = 1$ if (i,j) is the first pixel of a run, otherwise it is 0 (see Figure 1).

[1]Email: comic@uns.ac.rs. Research of L. Č. supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia through the project no. 451-03-68/2020-14/200156.
[2]Email: magillo@dibris.unige.it

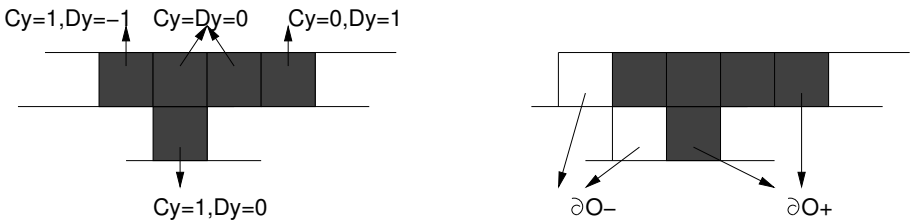


Figure 1: Left: the values of C_Y and D_Y for a run consisting of more pixels (top) or of just one pixel (bottom); only pixels starting or ending a run give a non-zero contribution to the moments. Right: black pixels belonging to ∂O^+ and white pixels belonging to ∂O^- in the same configuration.

An equivalent formulation of the discrete version of the Green's theorem, proposed by Philips [5], states that

$$\sum_{(i,j)\in O} \nabla_x f(i,j) = \sum_{(i,j)\in \partial O^+} f(i,j) - \sum_{(i,j)\in \partial O^-} f(i,j), \tag{4}$$

where $\nabla_x f(i,j) = f(i,j) - f(i-1,j)$ and ∂O^+ is the set of black pixels with white right neighbor, while ∂O^- is the set of white pixels with black right neighbor (see Figure 1).

2 Related Work

The relevant algorithms [3] either decompose the object into non-overlapping simple shapes (rectangles), or they use some form of the discrete Green's theorem.

2.1 Decomposition-Based Algorithms

Algorithms in this class are either designed for an image encoded in a specific data structure (quadtree [6] or run-length [8, 9, 13]) or they compute a decomposition in a pre-processing step [7, 10].

The δ-method proposed by Zakaria et al. [13] computes low-order moments of horizontally convex objects (i.e., with at most one run in each row). Li [4] generalized this algorithm to non-convex objects [2]. Spiliotis and Mertzios [8, 9] proposed another extension of [13], which first decomposes an object into disjoint rectangular blocks by merging consecutive runs of equal spread and then computes the discrete moments of arbitrary order on the rectangles. Sossa et al. [7] decompose the object into non-overlapping squares using morphological erosion. Their algorithm works also for objects with holes. Suk and Flusser [10] use distance transform to obtain the decomposition into squares.

The pre-processing stage, necessary to decompose the object, is expensive. Decomposition methods are convenient only if the object is compact (if it can be partitioned into a small number of squares) and a large number of moments is to be computed [10].

2.2 Boundary-Based Algorithms

Tang [11], Philips [5] and Yang and Albregtsen [12] proposed to use the discrete Green's theorem to compute low-order moments. In both Formulas (3) and (4), the only pixels giving a non-zero contribution to the sum are those lying on the contour of the object O, i.e, the pixels of O that are edge-adjacent to at least one white pixel.

The algorithm by Tang works on the cyclic sequence $(i_0, j_0), \ldots (i_l, j_l)$ of the contour pixels, which is given as a contour chain code. The used formula comes from (3) by taking $f(i,j) = i^p j^q$. This gives

$$F_x(i_n, j_n) = \sum_{h=0}^{i_n} f(h, j_n) = \sum_{h=0}^{i_n} h^p j_n^q = j_n^q S_p(i_n),$$

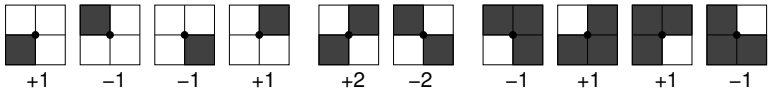


Figure 2: Various configurations defining the corner vertices. The shown coefficient is associated with the lower left pixel of the configuration.

and

$$\begin{aligned} m_{p,q}(O) = & \sum_{(i,j)\in O} f(i,j) = \sum_{(i,j)\in O} i^p j^q \\ = & \sum_{n=0}^{l-1} (F_x(i_n,j_n)D_Y(i_n,j_n) + f(i_n,j_n)C_Y(i_n,j_n)) \\ = & \sum_{n=0}^{l-1} (F_x(i_n,j_n)D_Y(i_n,j_n) + i_n^p j_n^q C_Y(i_n,j_n)). \end{aligned} \tag{5}$$

The algorithm by Philips [5] works on the runs, and applies the alternative formulation (4) of the discrete Green's theorem. The algorithm classifies the pixels during a raster scan. For the moment computation, Formula (4) is considered with $f(i,j) = g(i)j^q$, where $g(i)$ is such that $\nabla_x g(i) = i^p$. The moments are computed as

$$m_{p,q}(\mathrm{O}) = \sum_{(i,j)\in\partial O^+} S_p(i)j^q - \sum_{(i,j)\in\partial O^-} S_p(i)j^q, \tag{6}$$

where ∂O^+ is the set of end pixels of the runs, and the immediate right neighbors of white pixels in ∂O^- are start pixels of the runs (see Figure 1).

The algorithm by Yang and Albregtsen [12] considers the boundary of O (consisting of the edges between black and white pixels) instead of the contour of O (consisting of pixels). For horizontal edges, the term containing C_Y [11] vanishes. For vertical edges, $D_Y = \pm 1$ if the incident black pixel is the end pixel or the start pixel of a run, respectively. The final obtained formula is the same as that by Philips, but in [12] it is embedded within a contour following algorithm.

3 Our Formula

We suppose that the object O is in the first quadrant, and contained in a rectangle of size $N \times M$. That is, O is a set of (black) pixels with integer coordinates (i,j) with $1 \le i \le N$ and $1 \le j \le M$. We pose no restrictions on the configuration of black pixels, so the object does not need to be (simply) connected, or convex.

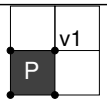
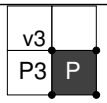
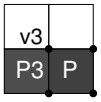
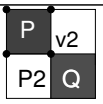
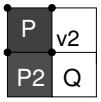
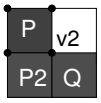
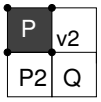
The coordinates of the four vertices of the pixel (i,j) are $(i \pm \frac{1}{2}, j \pm \frac{1}{2})$. We consider the boundary ∂O as a set of line segments at inter-pixel level. We define the corner vertices of O as those vertices v such that the four incident pixels of v are not all white or all black, and are not two edge-adjacent white pixels and two edge-adjacent black pixels. The configurations of corner vertices are shown in Figure 2.

We decompose the image into overlapping axis-aligned rectangles. Each rectangle is defined by the vertex $(\frac{1}{2}, \frac{1}{2})$ and by one corner vertex $(J + \frac{1}{2}, L + \frac{1}{2})$ of O. According to Formula (2), the moment of the rectangle associated with the corner vertex $(J + \frac{1}{2}, L + \frac{1}{2})$ is equal to $S_p(J) \cdot S_q(L)$.

The moments of O are then computed from rectangle moments through a simple inclusion-exclusion principle. We sum the moments of rectangles for each corner vertex in the boundary of O with the appropriate coefficient, shown in Figure 2. Our formula for moment computation can be summarized as:

$$m_{p,q}(O) = \sum_{(x,y)\in\partial O} V(x,y) \cdot S_p(x - \frac{1}{2}) \cdot S_q(y - \frac{1}{2}) \tag{7}$$

where $V(x,y)$, called corner code, is non-zero for corners only, and its values are shown in Figure 2. Note that $(x - \frac{1}{2}, y - \frac{1}{2})$ are the coordinates of the pixel having (x,y) as its upper right corner.

vert.	config.	contribution O	P	O'
v_1		1	1	0
v_3		-1	-1	0
v_3		0	-1	1
v_2		-2	-1	-1
v_2		0	-1	1
v_2		-1	-1	0
v_2		-1	-1	0
...	...		...	

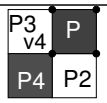
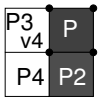
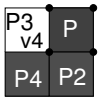
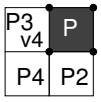
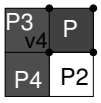
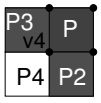
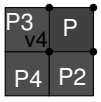
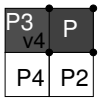
vert.	config.	contribution O	P	O'
v_4		2	1	1
v_4		0	1	-1
v_4		1	1	0
v_4		1	1	0
v_4		1	1	0
v_4		-1	-2	1
v_4		0	1	-1
v_4		0	1	-1

Figure 3: Inductive step of the proof. The contribution of each vertex v_1, v_2, v_3, v_4 to the moment of O, of P, and of $O' = O \setminus P$, in all possible configurations.

The proof of correctness of Formula 7 is by induction on the number k of pixels in O. In the inductive step, the contributions of the four vertices of the topmost leftmost pixel in O to the moments of O, of P, and of $O' = O \setminus P$ are evaluated, as illustrated in Figure 3. The contribution of the vertex v_l to $m^R_{p,q}(O)$ (first numeric column) is always equal to its contribution to $m^R_{p,q}(O') + m^R_{p,q}(P)$ (sum of the last two columns). Therefore $m^R_{p,q}(O') + m^R_{p,q}(P) = m^R_{p,q}(O)$ and thus (by induction) $m^R_{p,q}(O) = m_{p,q}(O)$.

4 Results and conclusions

We have proposed a simple inclusion-exclusion based formula for the computation of discrete geometric moments of 2D binary images. As opposed to many other decomposition-based formulas, ours is the only one based on the decomposition into overlapping rectangles. Our formula is especially suitable for images with boundary composed mainly of horizontal and vertical lines, and therefore having few corners. Objects with such characteristics are common products of computer art. As a test set, we consider 56 icons from `https://www.flaticon.com/free-icons/` (see Figure 4), and their rotated versions, obtained by exchanging rows and columns. On such 112 test inputs, we computed moments $m_{p,q}$ with $p + q \leq 3$ by using our proposed Formula (7), Formula (5) used by Tang and Formula (6) used by Philips. Since Formula (5) showed to be less efficient than (6), here we restrict the comparison to Formulas (7) and (6).

Figure 4: Test inputs for the experiments. Each icon was used twice, the second time with a rotation of 90 degrees. Icons are sorted from the best to the worst performance of our formula on them.

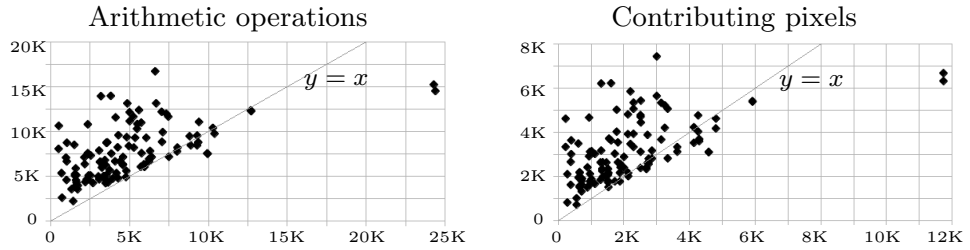


Figure 5: Comparison of Formula (6) by Philips versus our Formula (7). Left: the number of operations (additions and multiplications) in the computation of a single moment. Right: the number of pixels contributing to the computation of moments. The horizontal axis is our formula, the vertical one is Formula (6). Each dot is a tested image.

Figure 5 compares the number of pixels contributing to the moment computation (i.e., those with a non-zero coefficient), and the number of performed multiplications and additions, with Formulas (7) and (6). Most dots (precisely, 88 over 112) are above the bisecting line of the first quadrant, meaning that, on the corresponding test inputs, our formula performs better. The icons in Figure 4 are sorted from the one giving the best performance to the one giving the worst performance. This last icon and its rotated version are the two dots at the extreme right side of the plots in Figure 5.

We plan to extend this work to the computation of exact continuous geometric moments in 2D, as well as to images in arbitrary dimensions.

References

[1] L. Čomić and P. Magillo. "Computation of 2D Discrete Geometric Moments Through Inclusion-Exclusion". In: *Pattern Recognition and Artificial Intelligence - 3rd International Conference, ICPRAI*. Vol. 13296. LNCS. Springer, 2022.

[2] J. Flusser and T. Suk. *On the Calculation of Image Moments*. Tech. rep. 1946. Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, 1999.

[3] J. Flusser, T. Suk, and B. Zitova. *2D and 3D Image Analysis by Moments*. John Wiley & Sons, Ltd, 2016.

[4] B. C. Li. "A new computation of geometric moments". In: *Pattern Recognition* 26.1 (1993), pp. 109–113.

[5] W. Philips. "A new fast algorithm for moment computation". In: *Pattern Recognition* 26.11 (1993), pp. 1619–1621.

[6] M. Shneier. "Calculations of Geometric Properties Using Quadtrees". In: *Computer Graphics Image Processing* 16 (1981), pp. 296–302.

[7] J. H. Sossa-Azuela, C. Yáñez-Márquez, and J. L. Díaz-de-León S. "Computing geometric moments using morphological erosions". In: *Pattern Recognition* 34.2 (2001), pp. 271–276.

[8] I. M. Spiliotis and B. G. Mertzios. "Real-time computation of 2-D moments on block represented binary images on the scan line array processor". In: *8th European Signal Processing Conference, EUSIPCO*. 1996, pp. 1–4.

[9] I. M. Spiliotis and B. G. Mertzios. "Real-time computation of two-dimensional moments on binary images using image block representation". In: *IEEE Trans. Image Processing* 7.11 (1998), pp. 1609–1615.

[10] T. Suk and J. Flusser. "Refined Morphological Methods of Moment Computation". In: *20th International Conference on Pattern Recognition, ICPR*. 2010, pp. 966–970.

[11] G. Y. Tang. "A Discrete Version of Green's Theorem". In: *IEEE Trans. Pattern Anal. Mach. Intell.* 4.3 (1982), pp. 242–249.

[12] L. Yang and F. Albregtsen. "Fast computation of invariant geometric moments: a new method giving correct results". In: *12th IAPR International Conference on Pattern Recognition, Conference A: Computer Vision & Image Processing, ICPR, Volume 1*. 1994, pp. 201–204.

[13] M. F. Zakaria, L. J. Vroomen, P. J. Zsombor-Murray, and J. M. H. M. van Kessel. "Fast algorithm for the computation of moment invariants". In: *Pattern Recognition* 20.6 (1987), pp. 639–643.

Trees having domination number equal to $\{K_2\}$-isolation number

Adriana Dapena[1*], Magdalena Lemańska[2†], María José Souto-Salorio[3*], Francisco Vazquez-Araujo[4*]

[*]Universidade da Coruña. España
[†]Gdańsk University of Technology. Poland

Abstract

Let $T = (V_T, E_T)$ be a tree with $n =\mid V_T \mid \geq 3$ vertices. A subset $S \subseteq V_T$ is called dominating set if $V_T - N_T[S] = \emptyset$, where $N_T[S]$ denotes the closed neighborhood of the subset S. The minimum cardinality of a dominating set is the domination number and it is denoted by $\gamma(T)$. We say $W \subseteq V_T$ is an $\{K_2\}-$isolating set in T if the graph induced by $V_T - N_T[W]$ contains no edges. The minimum cardinality of a $\{K_2\}-$isolating set is the isolation number of T and it is denoted by $\iota(T)$. In this paper we give different equivalent characterizations of trees such that $\gamma(T) = \iota(T)$. Moreover, we focus our attention on trees that verify $\iota(T) = \frac{n}{3}$. We show they form a subfamily of those for which $\gamma(T) = \iota(T)$ holds.

1 Introduction

In [2], the authors introduce the definition of an $\mathcal{F}-$isolation set of a graph G for an arbitrary family of graphs $\mathcal{F}$. This notion gives a generalization of the classical domination problem. Indeed, $\{K_1\}-$isolating sets coincide with the usual dominating sets. This concept widely appears in designing communication networks. Describing the network by a graph, a dominating set D is such that all vertices outside have a neighbour inside. That is, if we delete the closed neighbourhood of D we get the graph with no vertices. Now, if we consider a $\{K_2\}-$isolating set W of G and delete the closed neighbourhood of W we obtain the graph with no edges.

In this work, we focus on $\mathcal{F}-$isolation numbers when $\mathcal{F} = \{K_i\}$ with $i = 1, 2$. Following [2, 1], the minimum cardinality of a $\{K_2\}-$isolating set of G it is denoted by $\iota(G)$ and is said the isolation number of G. The domination number as usual is denoted by $\gamma(G)$.

In general, the difference between $\iota(G)$ and $\gamma(G)$ can be arbitrarily large. In this paper, we focus on graphs for which both parameters are identical. When studying the equality between two parameters, very often the equality both in the graph and in all its induced subgraphs is considered. Graphs for which the two parameters a and b are equal both in the graph G itself and in all its induced subgraphs are called ab-perfect graphs. In [6] the authors gives algorithm test to decide whether a graph is a quasi-threshold graph and in [5] these graphs are identify with certain ab-perfect graphs. In our work, we begin showing that the $\gamma\iota$-perfect graphs coincide with the *quasi-threshold graphs.*

The main contribution of this work is the study of trees for which the two parameters coincide, that is $\iota(T) = \gamma(T)$. In [1] the authors prove that if G is a connected graph on $n \geq 3$ vertices and different from the cycle C_5 then $\iota(G) \leq \frac{n}{3}$ and this bound is sharp. This result motivates the study of the extremal trees such as $\iota(T) = n/3$. We prove that trees such as $\iota(T) = n/3$ form a subfamily of those for which $\gamma(T) = \iota(T)$. We also give some equivalent characterizations of all these trees. The full version of this work can be found in [4].

[1]Email: adriana.dapena@udc.es
[2]Email: magdalena.lemanska@pg.edu.pl
[3]Email: maria.souto.salorio@udc.es
[4]Email: francisco.vazquez@udc.es

2 Preliminaires

Let $G = (V_G, E_G)$ be a simple, undirected, finite graph with vertex set $V = V_G$ of order $n =\mid V \mid$ and edge set $E = E_G$. The open neighbourhood of a vertex $v \in V_G$, denoted $N_G(v)$, is the set of neighbours of v; thus, $N_G(v) = \{u \in V_G,\ uv \in E_G\}$. The closed neighborhood of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex $v \in V_G$, denoted by $d_G(v)$, is the number of neighbours of v, so $d_G(v) = |N_G(v)|$.

A leaf in a graph G is a vertex of degree 1 in G and a support vertex is a vertex adjacent to a leaf. A strong support vertex is adjacent to at least two leaves. The set of leaves of G is denoted by $\Omega(G)$ and the set of all support vertices of G is denoted by $Supp(G)$.

The subgraph of G induced by $S \subseteq V_G$ is denoted by $G[S]$ and the subgraph obtained from G by deleting all vertices in S and all edges incident with vertices in S is denoted by $G - S$.

We say that a set $X \subseteq V_G$ is a 2-packing if $d_G(u, v) > 2$ for any two vertices $u, v \in X$.

For a subset X of V_G, $N_G[X]$ denotes $\cup_{v \in X} N_G[v]$ the closed neighborhood of X.

A subset $S \subseteq V_G$ is called dominating in G if $N_G[S] = V_G$. The minimum cardinality of a dominating set of G is the domination number and is denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is called a minimum dominating set or $\gamma(G)$-set.

For short, and if there is no ambiguity, we will omit G in the notation for parameters.

In [1], A. Hansberg and Y. Caro introduce the following concept

Definition 1. Let G a simple graph and $\mathcal{F}$ a family of graphs. A set of vertices $S \subseteq V$ is said to be a $\mathcal{F}$-isolating set of G if the graph induced by the set $V - N_G[S]$ contains no member of $\mathcal{F}$ as a subgraph.

In particular, when $\mathcal{F} = \{K_2\}$ they use isolating set of G instead $\{K_2\}-$isolating set.

The minimum cardinality of a $\{K_2\}-$isolating set in G is the isolation number of G and it is denoted by $\iota(G)$. An isolating set of cardinality $\iota(G)$ is called a minimum isolating set or $\iota(G)$-set.

Note that $\{K_1\}-$isolating sets coincide with the usual dominating sets.

Remark 2. *In general, the domination number is bigger or equal than the isolation number. Moreover, the difference between $\iota(G)$ and $\gamma(G)$ can be arbitrarily large. For example,*

1. *Take the star $K_{1,r}$, with $r \geq 2$ leaves. We have $\iota(K_{1,r}) = \gamma(K_{1,r}) = 1$.*
2. *If we consider G the subdivided star, that is if every edge in the star $K_{1,r}$ was subdivided once, then $\gamma(G) = r$ and $\iota(G) = 1$.*
3. *If P_n is a path with $n \geq 3$ vertices then $\iota(P_n) = \lceil \frac{n-1}{4} \rceil$ and $\gamma(P_n) = \lceil \frac{n}{3} \rceil$.*

Now, we focus on $\gamma\iota$-perfect graphs that is graphs G for which the two parameters $\iota(G)$ and $\gamma(G)$ are equal both in the graph G itself and in all its induced subgraphs. First, we see the following observation that we will use before.

Remark 3. *In [6], quasi-threshold graphs are studied and several characterization of them are given. The author proved that quasi-threshold graphs are equivalent to P_4 and C_4-free graphs. That means, G is quasi-threshold graph if and if G does not contain neither P_4 nor C_4 as an induced subgraph. In particular, the class of quasi-threshold graphs is recursively defined as follows:*

1. *K_1 is a quasi-threshold graph;*
2. *from a quasi-threshold graph H we get another quasi-threshold graph G by adding an universal vertex v, that means $V_G = N_G[v]$;*
3. *disjoint union of quasi-threshold graphs.*

It follows that if G is quasi-threshold graph then $\iota(G) = \gamma(G) = 1$. In fact, in general, if a graph G has an universal vertex, then $\iota(G) = \gamma(G) = 1$.

Recall that perfect graphs coincide with $\omega\chi$-perfect graphs where ω denotes the clique number and χ the chromatic number. Quasi-threshold graphs are also known as trivially perfect graphs. In [5], the author proved that quasi-threshold graphs are the $\omega\lambda$-perfect graphs and also coincide with the $\lambda\chi$-perfect graphs where by λ we denote the pseudo-Grundy number.

The following result gives another characterization:

Theorem 4. *Let G be a connected graph. Then, G is a quasi-threshold if and only if G is $\gamma\iota$-perfect graph.*

Proof. If G is a quasi-threshold graph then it is $\gamma\iota$-perfect because quasi-threshold graphs have an universal vertex, what gives $l = \gamma = 1$.

By [6], G is a quasi-threshold graph if and only if P_4 and C_4-free graph. Using that $\gamma(P_4) = \gamma(C_4) = 2$ and $l(P_4) = l(C_4) = 1$ we infer that $\gamma\iota$-perfect graphs do not contain neither P_4 nor C_4 as an induced subgraph. We conclude that if G is a $\gamma\iota$-perfect graph then G is a quasi-threshold graph. □

3 Main result

In this section, we continue the study of the parameters γ and ι. We focus on trees graphs.

First, we characterize trees T for wich the equality between the two parameters $\gamma(T) = \iota(T)$ holds.

We begin by considering $\mathcal{T}$ the family consist of trees T that can be obtained from a star by recursively adding a new star as follows: T_{i+1} can be obtained from a sequence $T_1, \ldots, T_i$ $(i \geq 1)$ of trees such that:

- T_1 is a star $K_{1,r}$ $(r \geq 2)$ and,
- if $i \geq 1$, the tree T_{i+1} can be obtained from T_i by adding a star $K_{1,r}$ $(r \geq 2)$ and an edge xy, where x is a vertex at a distance two from a leaf of T_i and y is an end vertex of a star $K_{1,r}$.

The next theorem collects the equivalent characterizations of trees T with equality $\gamma(T) = \iota(T)$.

Theorem 5. *Let T be a tree of order $n \geq 3$. The following are equivalent:*

1. $T \in \mathcal{T}$.
2. *$Supp(T)$ is a $\gamma(T)$-set which forms a 2-packing of T.*
3. $|N_T[x] \cap Supp(T)| = 1$ *for every vertex* $x \in V_T$.
4. $\gamma(T) = \iota(T)$.

Proof. If $T \in \mathcal{T}$ then $Supp(T)$ forms a maximum 2-packing and it is the unique $\gamma(T)$-set not containing leaves. Moreover, if $P = (v_0, v_1, \ldots, v_m)$ is a longest path in T, we have $d_T(v_2) = 2$ and v_3 is at a distance 2 from a leaf. Thus, $T' = T - N_T[v_1]$ is a tree and we can obtain T from T' by attaching the star $N_T[v_1]$ and the edge v_2v_3. Therefore, $T' \in \mathcal{T}$ implies $T \in \mathcal{T}$. From this we can get that $1. \Leftrightarrow 2.$ and then 3.

Now, we prove $4. \Leftrightarrow 3.$ Assume $\gamma(T) = \iota(T) = l$ and take D be a minimum dominating set of T. Observe that if $x \in Supp(T)$ and there exists a vertex $x' \in N_T(x) \cap Supp(T)$ then $S_1 = D - \{x\}$ is an isolating set of T. If $x \notin Supp(T)$ and there are $s_1 \neq s_2$ two different support vertices such that $s_1, s_2 \in N_T(x)$ then $S_2 = (D - \{s_1, s_2\}) \cup \{x\}$ is an isolating set of T. In both cases we have a contradiction with $\gamma(T) = \iota(T)$. Then we have $\forall x \in Supp(T),\ N_T[x] \cap Supp(T) = \{x\}$ and $\forall x \notin Supp(T),\ |N_T[x] \cap Supp(T)| \leq 1$..

Now, by induction on $Supp$ we can prove that $\forall x \in V_T,\ |N_T[x] \cap Supp(T)| \geq 1$. In fact, if D is a minimum dominating set not containing leaves of T and $P = (v_0, \ldots, v_m)$ is a longest path of T then, $v_2 \notin Supp(T)$ and $d_T(v_2) = 2$. Then we can consider the tree $T' = T - N_T[v_1]$ and we conclude.

Now, if assume that for every $x \in V_T$ is $|N_T[x] \cap Supp(T)| = 1$ then also we get by induction on $s = |Supp(T)|$ that $Supp(T)$ is a minimum isolating set of T and $\gamma(T) = \iota(T) = s$. If $s = 1$, then T is a star and $\gamma(T) = \iota(T) = 1$. Assume the result holds for trees with $s' < s$ support vertices. Let T be a tree with $s = |Supp(T)| > 1$ support vertices. Take $P = (v_0, v_1, \ldots, v_m)$ a longest path in T and $T' = (T - N[v_1]) \cup \{v_2\}$. Then T' is a tree with $Supp(T') = Supp(T) - \{v_1\}$ and $|N[x] \cap Supp(T')| = 1$ for every vertex $x \in V_{T'}$. By induction, $Supp(T')$ is a minimum isolating set of T' and $s - 1 = \gamma(T') = \iota(T')$. We know $Supp(T')$ is a $\gamma(T')-$set. Then $Supp(T)$ is a $\gamma(T)-$set. Now, let W be a minimum isolating set of T. Therefore, the set $W \cap T'$ is an isolating set of T'. From the fact that $v_0, v_1 \in V_T - N_T[W \cap T']$ we have that $W \cap T'$ is not isolating set of T. Then $s - 1 = \iota(T') \leq |W \cap T'| < |W| = \iota(T) \leq \gamma(T) = s$ and we conclude that $\iota(T) = s = \gamma(T)$. □

Now, we investigate the extremal trees that belong to a subfamily of $\mathcal{T}$.
In [1] the authors prove that

Theorem 6. *If G is a connected graph on $n \geq 3$ vertices and different from the cycle C_5 then $\iota(G) \leq \frac{n}{3}$ and this bound is sharp.*

This result motivates the study of the trees such as $\iota(T) = \frac{n}{3}$. Related with this, we consider a special class of graphs from the family $\mathcal{T}$.

Definition 7. Graphs from the family $\mathcal{T}$ that we can build by taking only stars $K_{1,2}$ form a family that we denote by $\mathcal{R}$. That is, $\mathcal{R}$ is the family consist of trees T that can be obtained from a star $K_{1,2}$ by recursively adding a new star $K_{1,2}$ and an edge xy where x, y are leaves.

We have the following result.

Theorem 8. *Let T be a tree of order $n \geq 3$. The following are equivalent:*

1. $T \in \mathcal{R}$
2. $\gamma(T) = \iota(T) = \frac{n}{3}$
3. $\iota(T) = \frac{n}{3}$
4. *$Supp(T)$ is a 2-packing $\gamma(T)$-set with $\frac{n}{3}$ vertices.*

Proof. First, observe that if $T \in \mathcal{R}$ if and only if $T \in \mathcal{T}$ and $d_T(s) = 2$ for every $s \in Supp(T)$. Moreover, it follows that $\gamma(T) = \iota(T) = \frac{n}{3}$. Then we get *1.* implies *4.* and also *1.* implies *2.* implies *3.* By induction, we can get $\iota(T) = \frac{n}{3}$ if and only if $T \in \mathcal{R}$. □

References

[1] Y. Caro and A. Hansberg. "Partial domination—the isolation number of a graph". In: *Filomat* 31.12 (2017), pp. 3925–3944. DOI: 10.2298/fil1712925c.

[2] A. Hansberg and Y. Caro. "Isolation in graphs". In: *Electronic Notes in Discrete Mathematics* 50 (2015). LAGOS'15, pp. 465–470. DOI: https://doi.org/10.1016/j.endm.2015.07.077.

[3] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater. *Fundamentals of domination in graphs*. Vol. 208. Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1998, pp. xii+446.

[4] M. Lemańska, M. J. Souto-Salorio, A. Dapena, and F. J. Vazquez-Araujo. "Isolation Number versus Domination Number of Trees". In: *Mathematics* 9.12 (2021). DOI: 10.3390/math9121325.

[5] C. Rubio-Montiel. “A new characterization of trivially perfect graphs”. In: *Electron. J. Graph Theory Appl. (EJGTA)* 3.1 (2015), pp. 22–26. DOI: 10.5614/ejgta.2015.3.1.3.

[6] J.-H. Yan, J.-J. Chen, and G. J. Chang. “Quasi-threshold graphs”. In: *Discrete Appl. Math.* 69.3 (1996), pp. 247–255. DOI: 10.1016/0166-218X(96)00094-7.

Restrained differential of a graph

Magda Dettlaff[1*], Abel Cabrera-Martínez[2†], Magdalena Lemańska[3*], Juan Alberto Rodríguez-Velázquez[4‡]

[*]Gdańsk University of Technology, Poland
[†]University of Cordoba, Spain
[‡]Universitat Rovira y Virgili, Tarragona, Spain

1 Introduction

Given a graph $G = (V(G), E(G))$ and a vertex $v \in V(G)$, the open neighbourhood of v in G is defined to be $N_G(v) = \{u \in V(G) : uv \in E(G)\}$; when G is clear from the context, we write just $N(v)$ instead of $N_G(v)$. The open neighbourhood of a set $S \subseteq V(G)$ is defined as $N(S) = \cup_{v \in S} N(v)$, while the external neighbourhood of S, or boundary of S, is defined as $N_e(S) = N(S) \setminus S$. The differential of a set $S \subseteq V(G)$ is defined as $\partial(S) = |N_e(S)| - |S|$, while the differential of a graph G is defined to be

$$\partial(G) = \max\{\partial(S) : S \subseteq V(G)\}.$$

As described in [8], the definition of $\partial(G)$ was given by Hedetniemi about twenty five years ago in an unpublished paper, and was also considered by Goddard and Henning [6]. After that, the differential of a graph has been studied by several authors, including [2], [9], [12].

Given a set $S \subseteq V(G)$, let $S_r = \{v \in N_e(S) : N(v) \cap N_e(S) \neq \varnothing\}$. The *restrained differential of a set* $S \subseteq V(G)$ is defined as $\partial_r(S) = |S_r| - |S|$.

Here we introduce and study the *restrained differential* of a graph G, which is defined as

$$\partial_r(G) = \max\{\partial_r(S) : S \subseteq V(G)\}.$$

Among other results, we obtain general bounds on $\partial_r(G)$ and we prove a Gallai-type theorem, $\gamma_{rR}(G) + \partial_r(G) = n(G)$, where $\gamma_{rR}(G)$ denotes the restrained Roman domination number of G. The restrained Roman domination number is a well-known parameter, studied in [10], [7], [1] or [11].

Therefore, we can see that this novel parameter is perfectly integrated into the theory of domination. We can treat the theory of restrained differential in graphs as a new approach to the theory of restrained Roman domination. One of the advantages of this approach is that it allows us to study the restrained Roman domination number of a graph without the use of functions.

2 A Gallai-type theorem

A *Roman dominating function* (RDF) on a graph G is a function $f(V_0, V_1, V_2)$ such that for every vertex $v \in V_0$, there exists a vertex $u \in N(v) \cap V_2$. The *Roman domination number* of G, denoted by $\gamma_R(G)$, is the minimum weight $\omega(f) = \sum_{v \in V(G)} f(v)$ among all RDFs f on G.

A *restrained Roman dominating function* (RRDF) on a graph G is a RDF $f(V_0, V_1, V_2)$ such that the subgraph of G induced by V_0 has no isolated vertex, i.e., $\delta(G[V_0]) \geq 1$. The *restrained Roman domination number* of G, denoted by $\gamma_{rR}(G)$, is the minimum weight among all RRDFs f on G.

[1]Email: magda.dettlaff@pg.edu.pl
[2]Email: acmartinez@uco.es
[3]Email: magleman@pg.edu.pl
[4]Email: juanalberto.rodriguez@urv.cat

In general, a Gallai-type theorem has the form $a(G) + b(G) = n(G)$, where $a(G)$ and $b(G)$ are parameters defined on G and $n(G)$ is an order of G.

Theorem 1 (Gallai-type theorem for restrained differential)**.** *For any graph G,*

$$\gamma_{rR}(G) + \partial_r(G) = n(G).$$

As a consequence of this result and the well-known bounds $\gamma_r(G) \leq \gamma_{rR}(G) \leq 2\gamma_r(G)$ given in [10], where $\gamma_r(G)$ is a restrained domination number of G, we deduce the following theorem.

Theorem 2. *For any nontrivial graph G of order n,*

$$n - 2\gamma_r(G) \leq \partial_r(G) \leq n - \gamma_r(G).$$

3 General bounds

We next present some bounds on the restrained differential of a graph. In some cases we provide classes of graphs achieving the bounds, while in other cases we characterize the graphs reaching the equalities.

We say that a set $S \subseteq V(G)$ is a *2-packing* if $N[u] \cap N[v] = \varnothing$ for every pair of different vertices $u, v \in S$. A size of maximum 2-packing of G we call the 2-*packing number* of G and denote by $\rho(G)$. A maximum cardinality of an independent set of G we call an *independence number* of G and denote by $\alpha(G)$.

The following result improves the upper bound given in Theorem 2 for the graphs with minimum degree at least two.

Theorem 3. *For any graph G of order n with $\delta(G) \geq 2$,*

$$\partial_r(G) \leq n - \gamma_r(G) - \max\left\{\gamma_r(G) - \alpha(G), \frac{\gamma_r(G) + n(\delta(G) - 1)}{\delta(G)(\Delta(G) + 1)}\right\}.$$

Proposition 4. *If G is a graph of diameter three, then*

$$\partial_r(G) \geq \rho(G)(\delta(G) - 1).$$

As the next result shows, the bound above is tight.

Proposition 5. *If G is a δ-regular graph of order n with $\delta \geq 2$ and $\gamma(G) = \rho(G)$, where $\gamma(G)$ is a domination number of G, then*

$$\partial_r(G) = n - 2\rho(G) = \rho(G)(\delta - 1) = \frac{n(\delta - 1)}{\delta + 1}.$$

As an example of application of the result above, we consider the 3-cube graph, where $n = 8$, $\delta = 3$, $\rho(G) = 2$ and $\partial_r(G) = 4$.

Proposition 6. *If G is a claw-free graph of minimum degree at least two, then*

$$\partial_r(G) \geq \rho(G)(\delta(G) - 2).$$

From Theorem 2 we know that $n - 2\gamma_r(G) \leq \partial_r(G) \leq n - \gamma_r(G)$ for any nontrivial graph G of order n. The following result provides some new upper bounds for the restrained differential of G.

Proposition 7. *Let G be a nontrivial graph of order n. For any integer k such that $0 \leq k \leq \gamma_r(G)$,*

$$\partial_r(G) \leq \max\left\{n - 2\gamma_r(G) + k, \frac{(\Delta(G) - 1)(n - k - 1)}{\Delta(G) + 1}\right\}.$$

It is well-known ([2]) that $\partial_r(G) \leq \partial(G) \leq \frac{n(\Delta(G)-1)}{\Delta(G)+1}$ for any graph G of order n. We next provide an interesting equivalence.

Proposition 8. *Given a graph G of order n, $\partial_r(G) = \frac{n(\Delta(G)-1)}{\Delta(G)+1}$ if and only if $\gamma_r(G) = \frac{n}{\Delta(G)+1}$.*

The following result shows another family of graphs with $\partial_r(G) = \partial(G)$.

Theorem 9. *If G is a claw-free cubic graph, then $\partial_r(G) = \partial(G)$.*

4 Relations between restrained differential and other types of differentials

Currently, the study of differentials in graphs and their variants is of great interest because it has been observed that the study of different types of domination can be approached through a variant of the differential which is related to them. Specifically, we are referring to domination parameters that are necessarily defined through the use of functions, such as Roman domination, perfect Roman domination, Italian domination and unique response Roman domination. In each case, the main result linking the domination parameter to the corresponding differential is a Gallai-type theorem, which allows us to study these domination parameters without the use of functions. For instance, the differential ∂ is related to the Roman domination number γ_R the perfect differential ∂_p is related to the perfect Roman domination γ_R^p, the strong differential ∂_s is related to the Italian domination number γ_I and the 2-packing differential $\partial_{2\rho}$ is related to the unique response Roman domination number μ_R.

Perfect differential $\partial_p(G)$, strong differential $\partial_s(G)$ and 2-packing differential $\partial_{2\rho}(G)$ were introduced for any non-trivial graph G in [4], [5] and [3], respectively.

For these three differentials, similar like for restrained differential $\partial_r(G)$, Gallai-type theorems were proved, where these differentials were combined with perfect Roman domination number, Italian domination number and unique response Roman domination number, respectively.

Since for any graph G is $\gamma_I(G) \leq \gamma_R(G) \leq \gamma_R^p(G) \leq \mu_R(G)$ for Italian domination number γ_I, Roman domination number γ_R, pefrect Roman domination number γ_R^p and unique response Roman domination number $\mu_R(G)$, from these Gallai-type theorems we have the following:

Proposition 10. *For any connected graph G, $\partial_s(G) \geq \partial(G) \geq \partial_p(G) \geq \partial_{2\rho}(G)$.*

Let $\mathcal{G}_1$ be a family of graphs obtained from two copies of K_p joined by the edge for $p \geq 3$.

A tree containing exactly two support vertices u, v is called a double-star; if $d_G(u) = k + 1$ and $d_G(v) = l + 1$, then we donote such a graph by $S_{k,l}$. Let $\mathcal{G}_2$ be a family of double stars $DS_{p,p}$ for $p \geq 1$.

- if $G \in \mathcal{G}_1$, then $\partial(G) = \partial_r(G) = 2p - 4$ and $\partial_{2\rho}(G) = \partial_p(G) = p - 1$.
- if $G \in \mathcal{G}_2$, then $\partial_r(G) = 0, \partial(G) = \partial_p(G) = 2p - 2$ and $\partial_{2\rho}(G) = p$.

Since $\gamma_R(G) \leq \gamma_{rR}(G)$, we immediately obtain that $\partial(G) \geq \partial_r(G)$. There are some graphs G for which holds the equality $\partial_r(G) = \partial(G)$ (for example a graph from the family $\mathcal{G}_1$), but in general the difference between $\partial(G)$ and $\partial_r(G)$ can be arbitrarily large (see for example a graph from the family $\mathcal{G}_2$, where $\partial(G) - \partial_r(G) = 2p - 1$).

Notice that $\partial_r(G)$ and $\partial_{2\rho}$ (∂_p, respectively) are incomparable. If $G \in \mathcal{G}_1$, then $\partial_r(G) - \partial_{2\rho}(G) = p - 3$ ($\partial_r(G) - \partial_p(G) = p - 3$, respectively), while if $G \in \mathcal{G}_2$, then $\partial_{2\rho}(G) - \partial_r(G) = p$ ($\partial_p(G) - \partial_r(G) = p$, respectively).

5 Trees

Here we consider some bounds for the restrained differential in trees.

Theorem 11. *If T is a tree, then $\partial(T) = \partial_r(T)$ if and only if $T = K_1$ or $T = K_2$.*

Corollary 12. *If T is a tree with at least three vertices, then $\partial(T) \leq \partial_r(T) - 1$.*

Let $S(G) = S_1(G) \cup S_2(G)$ be the set of supports of G where $S_1(G)$ and $S_2(G)$ are the sets of weak and strong support of G, respectively. Let $\Omega(G)$ be the set of leaves of G.

Now we consider some lower bound for restrained differential of trees in terms of the order and the number of support vertices and leaves in a tree and we characterize the extremal trees. We introduce the family $\mathcal{R}$ of trees $T = T_i$ which can be obtained in the following way: Let $T_1 = P_4$. If i is a positive integer, then T_{i+1} can be obtained recursively from T_i by one of two operations:

Operation $\mathcal{O}_1$: if $v \in S(T_i)$, then we add a new vertex x and the edge xv.

Operation $\mathcal{O}_2$: if $v \in V(T_i)$ is such that $d(v, \Omega(T_i)) \equiv 0 \pmod 3$ and $N_{T_i}(v) \cap S_2(T_i) = \varnothing$, then we add a path $P_3 = (x, y, z)$ and the edge xv.

Theorem 13. *Let T be a tree of order $n \geq 3$, s support vertices and l leaves. Then $\partial_r(T) \geq (n-l+s-4)/3$ with equality if and only if $T \in \mathcal{R}$.*

The last result concerns the upper bound of restrained differential in terms of the order and the size of a maximum indepentent set $\alpha(T)$ of a tree T.

Theorem 14. *For any nontrivial tree T,*

$$\partial_r(T) \leq n(T) - \alpha(T) - 1.$$

References

[1] H. A. Ahangar and S. R. Mirmehdipour. "Bounds on the restrained Roman domination number of a graph". In: *Commun. Comb. Optim.* 1.1 (2016), pp. 75–82. DOI: 10.22049/CCO.2016.13556.

[2] S. Bermudo, J. C. Hernández-Gómez, J. M. Rodríguez, and J. M. Sigarreta. "Relations between the differential and parameters in graphs". In: *Conference on Discrete Mathematics and Computer Science (Spanish)*. Vol. 46. Electron. Notes Discrete Math. Elsevier Sci. B. V., Amsterdam, 2014, pp. 281–288. DOI: 10.1016/j.endm.2014.08.037.

[3] A. Cabrera Martínez, M. L. Puertas, and J. A. Rodríguez-Velázquez. "On the 2-packing differential of a graph". In: *Results Math.* 76.3 (2021), Paper No. 157, 24. DOI: 10.1007/s00025-021-01473-8.

[4] A. Cabrera Martínez and J. A. Rodríguez-Velázquez. "On the perfect differential of a graph". In: *Quaest. Math.* 45.3 (2022), pp. 327–345. DOI: 10.2989/16073606.2020.1858992.

[5] A. Cabrera Martínez and J. A. Rodríguez-Velázquez. *From the strong differential to Italian domination in graphs*. Submitted.

[6] W. Goddard and M. A. Henning. "Generalised domination and independence in graphs". In: *Congr. Numer.* 123 (1997), pp. 161–171.

[7] N. J. Krzywkowski M. Rad. *On the restrained Roman domination in graphs*. Mathematics (not published). 2004.

[8] J. L. Lewis, T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi, and P. J. Slater. "Differentials in graphs". In: *Util. Math.* 69 (2006), pp. 43–54.

[9] P. R. L. Pushpam and D. Yokesh. "Differentials in certain classes of graphs". In: *Tamkang J. Math.* 41.2 (2010), pp. 129–138.

[10] P. Roushini Leely Pushpam and S. Padmapriea. "Restrained Roman domination in graphs". In: *Trans. Comb.* 4.1 (2015), pp. 1–17.

[11] F. Siahpour, H. Abdollahzadeh Ahangar, and S. M. Sheikholeslami. "Some progress on the restrained Roman domination". In: *Bull. Malays. Math. Sci. Soc.* 44.2 (2021), pp. 733–751. DOI: 10.1007/s40840-020-00965-0.

[12] J. M. Sigarreta. "Differential in Cartesian product graphs". In: *Ars Combin.* 126 (2016), pp. 259–267.

Distance-equalizer sets of graphs

A. González[1*], C. Hernando[2†], M. Mora[3†]

[*]Dept. of Didactics of Mathematics, Universidad de Sevilla, Sevilla, Spain
[†]Dept. Matemàtiques, Universitat Politècnica de Catalunya, Barcelona, Spain

Abstract

A subset S of vertices of a connected graph G is a *distance-equalizer set* if for every two distinct vertices $x, y \in V(G) \setminus S$ there is a vertex $w \in S$ such that the distances from x and y to w are the same. The *equidistant dimension* of G is the minimum cardinality of a distance-equalizer set of G. This paper studies these sets and explores its properties and applications to other mathematical problems. Concretely, we first establish some bounds concerning the order, the maximum degree, the clique number and the independence number, and characterize all graphs attaining some extremal values. We then study distance-equalizer sets in several families of graphs. Finally, we show the usefulness of distance-equalizer sets for constructing doubly resolving sets.

1 Introduction

The notion of resolving set, also known as locating set, arises in diverse areas, including location problems in networks of different nature (see [3]). For example, in order to locate a failure in a computer network modeled as a graph G, we are interested in a subset of vertices S such that every vertex of the underlying graph is uniquely determined by its vector of distances to the vertices of S. Such a set is called a *resolving set* of the graph G, and the *metric dimension* of G, *dim(G)*, is the minimum cardinality of a resolving set.

Resolving sets and several related sets, such as identifying codes, locating-dominating sets, watching systems or doubly resolving sets, have been widely studied during the last decades (see [1, 2, 5, 9, 11, 12]). However, many recent papers [4, 8, 14] have turned their attention precisely in the opposite direction to resolvability, thus trying to study anonymization problems in networks instead of location aspects. For instance, the need to ensure privacy and anonymity in social networks makes necessary to develop graph tools. This paper studies new graph concepts that can also be applied to anonymization problems in networks: *distance-equalizer set* and *equidistant dimension*. Proofs are omitted because of limited space. A full version of this work can be found in [10].

The paper is organized as follows. In Section 2, we define distance-equalizer sets and the equidistant dimension, and show bounds in terms of other graph parameters: order, diameter, maximum degree, independence number, and clique number. Section 3 is devoted to characterize all graphs attaining some extremal values of the equidistant dimension. In Section 4 we study this parameter for some families of graphs: complete and complete multipartite graphs, bistars, paths, cycles, and Johnson graphs. For the particular cases of paths and cycles, we show that this parameter is related with 3-AP-free sets. In

[1]Email: gonzalezh@us.es. Research supported by "VI Plan Propio de Investigación y Transferencia" of the U.S. and by the Research Group in Mathematics Education FQM-226 of the Junta de Andalucía

[2]Email: carmen.hernando@upc.edu. Research supported by project PID2019-104129GB-I00 funded by MCIN/ AEI /10.13039/501100011033

[3]Email: merce.mora@upc.edu. Research supported by project PID2019-104129GB-I00 funded by MCIN/ AEI /10.13039/501100011033 and project H2020-MSCA-RISE project 734922 - CONNECT

Section 5, we obtain bounds for general graphs and trees on the minimum cardinality of doubly resolving sets in terms of the equidistant dimension. Finally, we present some open problems in Section 6.

All graphs considered in this paper are connected, undirected, simple and finite. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The *order* of G is $|V(G)|$. For any vertex $v \in V(G)$, its *open neighborhood* is the set $N(v) = \{w \in V(G) : vw \in E(G)\}$. The *degree* of a vertex v, denoted by $deg(v)$, is defined as the cardinality of $N(v)$. The *maximum degree* of G is $\Delta(G) = \max\{\deg(v) : v \in V(G)\}$ and its *minimum degree* is $\delta(G) = \min\{\deg(v) : v \in V(G)\}$. The *distance* between two vertices $v, w \in V(G)$ is denoted by $d(v, w)$, and the *diameter* of G is $D(G) = \max\{d(v, w) : v, w \in V(G)\}$. A *clique* is a subset of pairwise adjacent vertices and the *clique number* of G, denoted by $\omega(G)$, is the maximum cardinality of a clique of G. An *independent set* of G is a subset of pairwise non-adjacent vertices and the *independence number* of G, denoted by $\alpha(G)$, is the maximum cardinality of an independent set of G. We denote by P_n the *path* of order n, by C_n the *cycle* of order n, $n \geq 3$, and by K_n the *complete graph* of order n. For undefined terms we refer the reader to [15].

2 Distance-equalizer sets and equidistant dimension

Let x, y, w be vertices of a graph G. We say that w is *equidistant* from x and y if $d(x, w) = d(y, w)$. A subset S of vertices is called a *distance-equalizer set* for G if for every two distinct vertices $x, y \in V(G) \setminus S$ there exists a vertex $w \in S$ equidistant from x and y. The *equidistant dimension* of G, denoted by $eqdim(G)$, is the minimum cardinality of a distance-equalizer set of G. For example, if v is a universal vertex of a graph G, then $S = \{v\}$ is a minimum distance-equalizer set of G, and so $eqdim(G) = 1$.

The following results are immediate but make it easier to prove subsequent results.

Lemma 1. *Let G be a graph. If S is a distance-equalizer set of G and v is a support vertex of G, then S contains v or all leaves adjacent to v. Consequently, $eqdim(G) \geq |\{v \in V(G) : v \text{ is a support vertex}\}|$.*

Recall that a graph G is *bipartite* whenever $V(G)$ can be partitioned into two independent sets, say A, B, which are called its *partite sets*.

Proposition 2. *Let G be a bipartite graph with partite sets A and B. If S is a distance-equalizer set of G, then $A \subseteq S$ or $B \subseteq S$. Consequently, $eqdim(G) \geq min\{|A|, |B|\}$.*

If G is a graph of order n, with $n \geq 2$, then any set of vertices of cardinality $n - 1$ is obviously a distance-equalizer set. Hence, $n - 1$ is an immediate upper bound on the equidistant dimension of nontrivial graphs. We next prove some upper bounds involving classical graph parameters.

Proposition 3. *For every graph G of order $n \geq 2$, the following statements hold.*

i) $eqdim(G) \leq n - \Delta(G)$ and the bound is tight whenever $\Delta(G) \geq n/2$;

ii) $eqdim(G) \leq n - \omega(G)$, if $G \ncong K_n$, and the bound is tight;

iii) $eqdim(G) \leq \frac{n(D(G)-1)+1}{D(G)}$, and the bound is tight;

iv) $eqdim(G) \leq n - \alpha(G)$, whenever $D(G) = 2$, and the bound is tight.

The bound given in Proposition 3(i) is not tight for all values of $\Delta(G)$ and n, for example when $\Delta(G) = 2$ and $n \geq 7$. Indeed, the only graphs satisfying $\Delta(G) = 2$ are paths and cycles and, as it will be seen below, the equidistant dimension of paths and cycles of order $n \geq 7$ is at most $n - 3$.

3 Extremal values

In this section we characterize all nontrivial graphs achieving extremal values for the equidistant dimension, concretely, graphs G of order $n \geq 2$ such that $eqdim(G) \in \{1, 2, n-2, n-1\}$. We also derive a Nordhaus-Gaddum type bound for the equidistant dimension.

Theorem 4. *For every graph G of order $n \geq 2$, the following statements hold.*

i) $eqdim(G) = 1$ *if and only if* $\Delta(G) = n - 1$*;*

ii) $eqdim(G) = 2$ *if and only if* $\Delta(G) = n - 2$*.*

Theorem 5. *For any graph G of order n, the following statements hold.*

i) If $n \geq 2$, then $eqdim(G) = n - 1$ if and only if G is a path of order 2.

ii) If $n \geq 3$, then $eqdim(G) = n - 2$ if and only if $G \in \{P_3, P_4, P_5, P_6, C_3, C_4, C_5\}$.

Corollary 6. *If G is a graph of order $n \geq 7$, then $1 \leq eqdim(G) \leq n - 3$.*

Now, we provide a Nordhaus-Gaddum type bound on the equidistant dimension. Nordhaus-Gaddum type inequalities establish bounds on the sum of a parameter for a graph and its complement. Recall that the *complement* of a graph G, denoted by $\overline{G}$, is the graph on the same vertices as G and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Also, a graph G is *doubly connected* if both G and $\overline{G}$ are connected. Note that nontrivial doubly connected graphs have order at least 4.

Proposition 7. *If G is a doubly connected graph, then $eqdim(\overline{G}) \leq \delta(G) + 1$.*

Theorem 8. *If G is a doubly connected graph of order $n \geq 4$, then*

$$4 \leq eqdim(G) + eqdim(\overline{G}) \leq n + 1.$$

Moreover, these bounds are tight.

4 Equidistant dimension of some families of graphs

In this section we study the equidistant dimension of some families of graphs, concretely of complete, complete bipartite and complete multipartite graphs, bistars, paths, cycles, and Johnson graphs.

Theorem 9. *Let $n, r, s, p, n_1, \ldots, n_p$ be positive integers such that $n \geq 2$, $s \geq r$, $p \geq 3$ and $n_p \geq \cdots \geq n_1 \geq 1$. Then, the following statements hold.*

i) $eqdim(K_n) = 1$*;*

ii) $eqdim(K_{r,s}) = r$*;*

iii) $eqdim(K_{n_1,\ldots,n_p}) = \min\{n_1, 3\}$*;*

iv) $eqdim(K_2(r, s)) = r$

Distance-equalizer sets and the equidistant dimension of paths are related with 3-AP-free sets and the function $r(n)$ introduced by Erdös and Turán [7]. For every integer $n \geq 1$, let $[n] = \{1, 2, \ldots, n\}$. A subset $S \subseteq [n]$ is 3-*AP-free* if $a + c \neq 2b$, for every distinct terms $a, b, c \in S$ [13]. The largest cardinality of a 3-AP-free subset of $[n]$ is denoted by $r(n)$. We begin by introducing some preliminary results. A subset of $[n]$ is called *even-sum* if all its elements have the same parity.

Proposition 10. *For every positive integer n, it holds that*

$$eqdim(P_n) = n - max\{\, |T| \,:\, T \text{ is a 3-AP-free even-sum subset of } [n]\}.$$

Proposition 11. *[6] Let $k_1, ..., k_r, n$ be different positive integers. Then, one of the sets $\{2k_1 - 1, 2k_2 - 1, ..., 2k_r - 1\}$ or $\{2k_1, 2k_2, ..., 2k_r\}$ is a 3-AP-free even-sum set of $[n]$ if and only if $\{k_1, ..., k_r\}$ is a 3-AP-free subset of $\left[\, \lceil n/2 \rceil \,\right]$.*

The equidistant dimension of a path is derived from the results above.

Theorem 12. *For every positive integer n, it holds that*

$$eqdim(P_n) = n - r\left(\left\lceil \frac{n}{2} \right\rceil\right).$$

The equidistant dimension of cycles of even order is completely determined, while for cycles of odd order, lower and upper bounds in terms of $r(n)$ are given.

Theorem 13. *For every positive integer $n \geq 3$, the following statements hold.*

i) $eqdim(C_n) = \begin{cases} \frac{n}{2}, & \text{for } n \text{ even}, \ n \not\equiv 0 \mod 4; \\ \frac{3n}{4} - 1, & \text{for } n \text{ even}, \ n \equiv 0 \mod 4. \end{cases}$

ii) $\frac{n-1}{2} \leq eqdim(C_n) \leq n - r\left(\left\lceil \frac{n+1}{4} \right\rceil\right)$, *for n odd.*

Johnson graphs are important because of their connections with other combinatorial structures such as projective planes and symmetric designs [1]. Furthermore, there exist different studies about geometric versions of these graphs because of their multiple applications in network design. Due to these facts, among others, properties of Johnson graphs have been widely studied in the literature. We study the equidistant dimension of Johnson graphs, obtaining an upper bound for several cases.

The *Johnson graph* $J(n, k)$, with $n > k \geq 1$, has as vertex set the k-subsets of an n-set and two vertices are adjacent if their intersection has size $k - 1$. Thus, it can be easily seen that the distance between any two vertices X, Y is given by $d(X, Y) = |X \setminus Y| = |Y \setminus X| = k - |X \cap Y|$. Consequently, a vertex $U \in V(J(n, k))$ is equidistant from vertices X and Y if and only if $|U \cap X| = |U \cap Y|$.

Proposition 14. *For any positive integer k, it holds that $eqdim(J(n, k)) \leq n$ whenever $n \in \{2k - 1, 2k + 1\}$ or $n > 2k^2$.*

5 Using distance-equalizer sets for constructing doubly resolving sets

Doubly resolving sets were introduced in [3] as a tool for computing the metric dimension of cartesian products of graphs. Furthermore, different authors have provided interesting applications of doubly resolving sets on source location, algorithmic studies and relations with other graph parameters. We say that two vertices u, v *doubly resolve* a pair of vertices x, y of G (or that $\{x, y\}$ is *doubly resolved* by u, v) if $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. A set $S \subseteq V(G)$ is a *doubly resolving set* of G if every pair $\{x, y\} \subseteq V(G)$ is doubly resolved by two vertices of S (it is said that S *doubly resolves* $\{x, y\}$), and the minimum cardinality of such a set is denoted by $\psi(G)$. Observe that a doubly resolving set is also a resolving set, and so $dim(G) \leq \psi(G)$.

Proposition 15. *For every graph G, it holds that $\psi(G) \leq dim(G) + 2\, eqdim(G)$.*

Theorem 16. *For every tree T, it holds that $\psi(T) \leq dim(T) + eqdim(T)$.*

We finish this section by analyzing lower and upper bounds on $dim(G) + eqdim(G)$. Concretely, we are interested in the minimum and maximum value of $dim(G) + eqdim(G)$ for graphs of order n. First, note that for any nontrivial graph G of order n,

$$2 \leq dim(G) + eqdim(G) \leq 2(n-1). \tag{1}$$

The lower bound in (1) is attained only by the paths P_2 and P_3, by Theorem 4(i), and the upper bound, only by the path P_2, by Theorem 5(i). Hence, for every graph G of order at least 4,

$$3 \leq dim(G) + eqdim(G) \leq 2n - 3.$$

In order to study this question, we consider the following functions defined for integers $n \geq 4$:

$$\Sigma(n) := \max\{dim(G) + eqdim(G) \ : \ |V(G)| = n\}$$
$$\sigma(n) := \min\{dim(G) + eqdim(G) \ : \ |V(G)| = n\}.$$

Proposition 17. *For every integer $n \geq 4$, the following statements hold.*

i) $\Sigma(n) \geq \frac{3n}{2} - 3$;

ii) $\sigma(n) \leq \log_2(n) + 2$.

6 Open problems

We do not know the exact value of the equidistant dimension of trees. However, in this family of graphs, it looks that paths are those graphs needing more vertices to construct a distance-equalizer set. Indeed, it is easily seen that, for every pair of vertices of a path, there is at most one equidistant vertex. Hence, we believe that the following conjecture holds.

Conjecture 18. *If T is a tree of order n, then $eqdim(T) \leq eqdim(P_n)$.*

We also propose the following conjecture, that we prove in Theorem 16 that it holds true for trees.

Conjecture 19. For every graph G, it holds that $\psi(G) \leq dim(G) + eqdim(G)$.

References

[1] R. F. Bailey, J. Cáceres, D. Garijo, A. González, A. Márquez, K. Meagher, and M. L. Puertas. "Resolving sets for Johnson and Kneser graphs". In: *European J. Combin.* 34.4 (2013), pp. 736–751. DOI: 10.1016/j.ejc.2012.10.008.

[2] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, and M. L. Puertas. "On the metric dimension of infinite graphs". In: *Discrete Appl. Math.* 160.18 (2012), pp. 2618–2626. DOI: 10.1016/j.dam.2011.12.009.

[3] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, and D. R. Wood. "On the metric dimension of Cartesian products of graphs". In: *SIAM J. Discrete Math.* 21.2 (2007), pp. 423–441. DOI: 10.1137/050641867.

[4] J. Casas-Roma, J. Herrera-Joancomartí, and V. Torra. "A survey of graph-modification techniques for privacy-preserving on networks". In: *Artificial Intelligence Review* 47.3 (2017), pp. 341–366. DOI: 10.1007/s10462-016-9484-8.

[5] M. Claverol, A. García, G. Hernández, C. Hernando, M. Maureso, M. Mora, and J. Tejel. "Metric dimension of maximal outerplanar graphs". In: *Bull. Malays. Math. Sci. Soc.* 44.4 (2021), pp. 2603–2630. DOI: 10.1007/s40840-020-01068-6.

[6] E. J. Cockayne and S. T. Hedetniemi. "On the diagonal queens domination problem". In: *J. Combin. Theory Ser. A* 42.1 (1986), pp. 137–139. DOI: 10.1016/0097-3165(86)90012-9.

[7] P. Erdös and P. Turán. "On Some Sequences of Integers". In: *J. London Math. Soc.* 11.4 (1936), pp. 261–264. DOI: 10.1112/jlms/s1-11.4.261.

[8] T. Feder, S. U. Nabar, and E. Terzi. *Anonymizing Graphs*. Preprint. 2008. DOI: 10.48550/ARXIV.0810.5578.

[9] A. González, C. Hernando, and M. Mora. "Metric-locating-dominating sets of graphs for constructing related subsets of vertices". In: *Appl. Math. Comput.* 332 (2018), pp. 449–456. DOI: 10.1016/j.amc.2018.03.053.

[10] A. González, C. Hernando, and M. Mora. "The Equidistant Dimension of Graphs". In: *Bulletin of the Malaysian Mathematical Sciences Society* (2022). To appear. DOI: 10.1007/s40840-022-01295-z.

[11] C. Hernando, M. Mora, and I. M. Pelayo. "Watching systems in complete bipartite graphs". In: *VIII Jornadas de Matemática Discreta y Algorítmica*. Editorial Universidad de Almería, 2012, pp. 53–60.

[12] C. Hernando, M. Mora, I. M. Pelayo, C. Seara, and D. R. Wood. "Extremal graph theory for metric dimension and diameter". In: *Electron. J. Combin.* 17.1 (2010), Research Paper 30, 28.

[13] A. Sharma. "Sequences of integers avoiding 3-term arithmetic progressions". In: *Electron. J. Combin.* 19.1 (2012), Paper 27, 4.

[14] R. Trujillo Rasua and I. G. Yero. "k-Metric Antidimension: a Privacy Measure for Social Graphs". In: *Information Sciences* 328 (2016), pp. 403–417.

[15] D. B. West. *Introduction to graph theory*. Second Edition, Prentice Hall, 2001, pp. xvi+512.

The discrete side of Distance Geometry: a focus on the 1-dimensional case

Simon B. Hengeveld[1], A. Mucherino[2]

IRISA, University of Rennes 1, Rennes, France

This research is supported by the ANR-PRCI multiBioStruct *project (ANR-19-CE45-0019).*

Abstract

We review some recent results in the context of Distance Geometry in dimension 1, and give some new research directions for problem instances where the distance information is imprecise and represented by real-valued intervals.

1 Distance Geometry in dimension 1

If a sketch comedy were to introduce the problem we consider, it would probably show a few improvised robbers meeting with the gang boss in an attempt to get ready for their *mission*. You would then hear the boss shout "Synchronize your watches", and hence listen the robbers give diverging answers for the current time... until the last robber, after a little hesitation, would finally admit that his watch has since long been broken[3]...

Try it yourself with some friends and you will see that the situation above is not so uncommon (exception made for the broken watch), for in our everyday life it is not necessary to be extremely precise with time. The problem of synchronizing clocks with high precision is rather a problem that computer machines need more often, and more importantly, deal with. Over the Internet, for example, several messages are sent and received each second by the machines connected to the Net, and, as pointed out by Lamport in the seminal paper [3], the *logical nature* of time is of primary importance when designing or analyzing distributed systems. Since then, several research lines have developed in this context [1, 11, 12, 13].

Distance Geometry (DG) is a more general problem that is able to model the clock synchronization problem when its dimensionality is set to 1 [7]. We are given a simple weighted undirected graph $G = (V, E, d)$ whose edges represent the available distance information between vertex pairs, and the main problem in DG is to find out what are the positions we can assign to the vertices of the graph G so that the available distances are satisfied. These positions can, in general, be given in any dimension $K > 0$. In the context of clock synchronization, the vertices $v \in V$ are computer machines, the presence of an edge $\{u, v\} \in E$ between two machines indicates that a time offset is available, and the weight function d provides the numerical value for such an offset. And as mentioned above, K is set to 1. Saxe proved in 1979, in the context of graph embedabbility, that the problem is NP-complete [9] when $K = 1$.

In this short contribution, we will review some of our most recent results in the context of DG in dimension 1, and we will give some new directions for extending these results to the case where the distances are represented by real-valued intervals (instead of singletons).

[1]Email: simon.hengeveld@irisa.fr

[2]Email: antonio.mucherino@irisa.fr

[3]One of us remembers having seen this comedy sketch in a movie, probably an Italian comedy of the 80s; however, it is likely that the same comedy sketch was reproposed in other places.

2 Paradoxical Distance Geometry

The *paradoxical* subclass of DG instances in dimension 1 has recently been introduced in [2]. Instances belonging to this special class have the particularity of being on the one hand "easy" to solve because most vertices in G depend on only one other vertex, and the total number of expected solutions may be known in advance [5]; on the other hand, however, these instances admit a search space that grows exponentially with the cardinality of V, and its complete exploration is generally necessary in order to identify the solutions. Thus, these are rather "hard" instances.

Paradoxical instances of the DG are represented in dimension 1 by cycle graphs. Once the vertex to play the role of *first* vertex v_1 is selected, and one of its two neighbours is selected to play the role of the *second* vertex, say v_2, then a total vertex order on V can be derived. The first vertex can be simply placed in the origin of the real line $\Re$. Subsequently, the positions for every vertex v_k, with $k > 1$, can be obtained by exploiting the distance information related to its predecessor v_{k-1}.

If we suppose that all available distances are *exact*, which is, extremely precise, then, for every known position x_{k-1} for v_{k-1}, we can compute two new possible positions, $x_{k-1} - d_{k-1,k}$ and $x_{k-1} + d_{k-1,k}$, for the vertex v_k. We remark that this procedure builds up a binary tree whose paths, from the root to any of its leaf nodes, represent possible solutions to our problem [4]. Each position x_k is computed by performing partial sums of the distance values, each taken with the sign associated to the tree branches of the considered path. The feasible solutions are those for which the only "extra distance", i.e. the only one that is not employed for the computation of the positions and is associated to the edge $\{1, n\}$, is satisfied by the positions finally assigned to v_n. It is only when the last layer of the tree is reached, indeed, that the feasible solutions can be selected from the search space.

As pointed out in [2], the introduction of a fictive vertex v_{n+1} in G, artificially connected to v_n by the edge $\{n, n+1\}$ holding the same weight as $\{1, n\}$, together with the suppression of the original edge $\{1, n\}$, allows us to have a uniform distribution of the distances over the vertices of the graph. The immediate consequence of this transformation consists in the possibility of *blindly* applying the same rule over the vertices of the graph, in the derived vertex order, for the computation of all vertex positions. After the graph transformation, the selection of the feasible solutions from the search space can be performed by verifying whether the positions for v_1 and v_{n+1} coincide (this is a valid solution), or not (the solution is not valid). An important consequence of this transformation is the matrix-by-vector reformulation that we will describe in Section 3.

In order to establish the high importance of this special class of instances, we remark that every DG instance can be seen as the union of paradoxical instances [8]. A solution method capable to recognize, extract and solve with priority the paradoxical parts of DG instances could in fact benefit of better convergence properties: once the paradoxical sub-instances are solved, they can be replaced by their relatively tiny solution sets, aiding in this way at reducing the degrees of freedom for the solutions of the original instance.

3 Matrix-by-vector reformulation

The procedure above for the solution of paradoxical DG instances is based on the idea to build up a binary tree containing vertex positions and to select the feasible solutions by verifying that $x_1 = x_{n+1}$. We introduce the matrix[4]

$$M_{ij} = \begin{cases} -1 & \text{if } (i-1)/2^{j-1} \bmod 2 = 0, \\ 1 & \text{otherwise,} \end{cases}$$

and the vector $y_j = d_{j,j+1}$, which contains the distance information related to our paradoxical instance. Notice that the index i varies from 1 to 2^n, while the index j varies from 1 to n. It can be easily

[4]We can point out here again the paradoxical side of our instances: the multiplication of a matrix by a vector is nothing difficult to perform; however, we remark that the number of rows of our matrices grow exponentially with the number of vertices in G, so that very large matrices may be defined for relatively small instances.

verified [2] that the vector

$$r = My, \tag{1}$$

resulting from the multiplication of the matrix M by the vector y, contains the positions x_{n+1} for all possible solutions. The feasible solutions to our paradoxical instances are therefore the ones for which $r_i = 0$ (because x_1 is implicitly set to 0). Moreover, for a given matrix row i, the value of each x_k^i (which is the position for the vertex v_k in the solution encoded by the i^{th} row of the matrix) can be obtained by performing the partial sums $\sum_{j=1}^{k} M_{ij} y_j$. We point out that this matrix-by-vector reformulation is particularly adapted to the optical processor presented in [2], and that similar approaches have also been proposed for other well-known NP-complete problems [10].

4 Generalization to interval distances

An approach to deal with DG instances in dimension 1 where distances are represented by real-valued intervals was previously proposed in [7]. The main idea is to replace standard arithmetic with interval arithmetic for the calculation of possible subsets of interval values for the vertices of the graph G. Basically, only the addition and subtraction operations need to be borrowed from interval arithmetic. We notice that interval arithmetic has already been successfully employed in the context of global optimization [6], and that, in our context, it is particularly useful for modeling distance uncertainty.

Let us suppose that the graph $G = (V, E, d)$ represents a paradoxical DG instance where the distance information is provided through real-valued intervals, which is, the weight function d actually provides a lower bound ℓ and an upper bound u on each distance value. Let z be a vector consisting of n intervals, where $n = |V|$. Once the degenerate interval $\{0\}$ is associated to the first element z_1 of the vector, the subsequent intervals z_k can be obtained via the formulæ:

$$\begin{aligned} [\underline{z}_k, \bar{z}_k]' &= [\underline{z}_{k-1}, \bar{z}_{k-1}] - [\ell_{k-1,k}, u_{k-1,k}] = [\underline{z}_{k-1} - u_{k-1,k},\ \bar{z}_{k-1} - \ell_{k-1,k}]\,, \\ [\underline{z}_k, \bar{z}_k]'' &= [\underline{z}_{k-1}, \bar{z}_{k-1}] + [\ell_{k-1,k}, u_{k-1,k}] = [\underline{z}_{k-1} + \ell_{k-1,k},\ \bar{z}_{k-1} + u_{k-1,k}]\,, \end{aligned}$$

where $\underline{z}$ and $\bar{z}$ are the two vectors of real numbers containing, respectively, the lower and the upper bounds of the intervals in z. When the fictive vertex v_{n+1} is employed, the verification of feasibility for the obtained solutions reduces to verifying that $[\underline{z}_{n+1}, \bar{z}_{n+1}] \cap [\underline{z}_1, \bar{z}_1] \neq \emptyset$, which is equivalent to saying that the last interval z_{n+1} needs to contain the origin 0 (when z_1 is set up as indicated above).

But how to compute all intervals for the vector z? We claim in this short contribution that the use of interval distances in y, coupled with the use of interval arithmetic in Equ. (1), naturally extends the computations for the interval case. However, when classical arithmetic is replaced by interval arithmetic, the range of the computed intervals tends to increase more and more for larger indices k. As a consequence, a larger and larger uncertainty is likely to be associated to the possible values of the vertex positions as our instances grow in size.

The matrix-by-vector formulation, however, when employed for the interval case, allows us to rewrite in a straightforward manner the *back-propagation* procedure previously introduced in [7]. This procedure aims to reduce the range of the obtained intervals z_k by back-propagating any additional distance information that may be available for the computation of intervals z_h with $h > k$. When considering paradoxical DG instances, this situation only occurs when positions for the fictive vertex v_{n+1} are computed, which are supposed to be compatible with the position assigned to v_1 (because of the original edge $\{1, n\}$).

Let S be the subset of indices for which the corresponding row in M represents a feasible solution:

$$S = \{i \in \{1, \ldots, n\} \,:\, z_1 \cap [\underline{r}_i, \bar{r}_i] \neq \emptyset\}\,.$$

We claim that the actual position intervals z_k^i, for every feasible solution with index $i \in S$ and every vertex v_k of our graph G, are given by the following interval arithmetic formula:

$$[\underline{z}_k^i, \bar{z}_k^i] = \left(\sum_{j=1}^{k} M_{ij}[\underline{y}_j, \bar{y}_j]\right) \cap \left(\sum_{j=n}^{k} M_{ij}[\underline{y}_j, \bar{y}_j]\right).$$

The level of uncertainty on these intervals can therefore be reduced for all vertex positions having a very small, or a very large (close to n), rank in the considered vertex order. From the vector z, a solution x formed by singletons can be trivially extracted. Future works will be aimed at extending the present work to DG paradoxical instances in higher dimensions.

Acknowledgments

This work is partially supported by the international project MULTIBIOSTRUCT funded by the ANR French funding agency (ANR-19-CE45-0019).

References

[1] N. M. Freris, S. R. Graham, and P. R. Kumar. "Fundamental limits on synchronizing clocks over networks". In: *IEEE Trans. Automat. Control* 56.6 (2011), pp. 1352–1364. DOI: 10.1109/TAC.2010.2089210.

[2] S. B. Hengeveld, N. Rubiano da Silva, D. S. Gonçalves, P. Souto Ribeiro, and A. Mucherino. "An optical processor for matrix-by-vector multiplication: an application to the distance geometry problem in 1D". In: *Journal of Optics* 24.1 (2021), p. 015701. DOI: 10.1088/2040-8986/ac3a9e.

[3] L. Lamport. "Time, Clocks and the Ordering of Events in a Distributed System". In: *Communications of the ACM* (1978), pp. 558–565.

[4] L. Liberti, C. Lavor, and N. Maculan. "A branch-and-prune algorithm for the molecular distance geometry problem". In: *Int. Trans. Oper. Res.* 15.1 (2008), pp. 1–17. DOI: 10.1111/j.1475-3995.2007.00622.x.

[5] L. Liberti, B. Masson, J. Lee, C. Lavor, and A. Mucherino. "On the number of realizations of certain Henneberg graphs arising in protein conformation". In: *Discrete Appl. Math.* 165 (2014), pp. 213–232. DOI: 10.1016/j.dam.2013.01.020.

[6] F. Messine. "Extensions of affine arithmetic: application to unconstrained global optimization". In: *J.UCS* 8.11 (2002), pp. 992–1015.

[7] A. Mucherino. "On the exact solution of the distance geometry with interval distances in dimension 1". In: *Recent advances in computational optimization*. Vol. 717. Stud. Comput. Intell. Springer, Cham, 2018, pp. 123–134. DOI: 10.1007/978-3-319-59861-1_8.

[8] A. Mucherino. "An Analysis on the Degrees of Freedom of Binary Representations for Solutions to Discretizable Distance Geometry Problems". In: *Recent Advances in Computational Optimization: Results of the Workshop on Computational Optimization WCO 2020*. Ed. by S. Fidanova. Vol. 986. Cham: Springer International Publishing, 2022, pp. 251–255. DOI: 10.1007/978-3-030-82397-9_13.

[9] J. Saxe. "Embeddability of Weighted Graphs in K-space is Strongly NP-hard". In: *Proceedings of 17th Allerton Conference in Communications, Control and Computing*. CMU-CS-80-102. Carnegie-Mellon University, Department of Computer Science, 1979, pp. 480–489.

[10] N. T. Shaked, S. Messika, S. Dolev, and J. Rosen. "Optical solution for bounded NP-complete problems". In: *Appl. Opt.* 46.5 (2007), pp. 711–724. DOI: 10.1364/AO.46.000711.

[11] A. Singer. "Angular synchronization by eigenvectors and semidefinite programming". In: *Appl. Comput. Harmon. Anal.* 30.1 (2011), pp. 20–36. DOI: 10.1016/j.acha.2010.02.001.

[12] P. Veríssimo and M. Raynal. "Time in Distributed System Models and Algorithms". In: *Advances in Distributed Systems: Advanced Distributed Computing: From Algorithms to Systems*. Ed. by S. Krakowiak and S. Shrivastava. Berlin, Heidelberg: Springer Berlin Heidelberg, 2000, pp. 1–32. DOI: 10.1007/3-540-46475-1_1.

[13] Y.-C. Wu, Q. Chaudhari, and E. Serpedin. "Clock Synchronization of Wireless Sensor Networks". In: *IEEE Signal Processing Magazine* 28.1 (2011), pp. 124–138. DOI: 10.1109/MSP.2010.938757.

On maximum independent sets of almost bipartite graphs

Vadim E. Levit[1*], Eugen Mandrescu[2†]

[*]Dept. of Mathematics, Ariel University, Israel
[†]Dept. of Computer Science, Holon Institute of Technology, Israel

Abstract

The *independence number* $\alpha(G)$ is the cardinality of a maximum independent set, while $\mu(G)$ is the size of a maximum matching in the graph $G=(V,E)$. If $\alpha(G)+\mu(G)$ equals the order of G, then G is a *König-Egerváry graph* [4, 18]. The number $d(G)=\max\{|A|-|N(A)|: A\subseteq V\}$ is called the *critical difference* of G [19] (where $N(A)=\{v: v\in V, N(v)\cap A\neq\emptyset\}$). A set $X\subseteq V$ is *critical* if $|X|-|N(X)|=d(G)$.

A graph G is *unicyclic* if it has a unique cycle and *almost bipartite* if it has only one odd cycle.

Let $\ker(G)=\bigcap\{S: S \textit{ is a critical independent set}\}$, $\text{core}(G)$ be the intersection of all maximum independent sets, and $\text{corona}(G)$ be the union of all maximum independent sets. It is known that $\ker(G)\subseteq\text{core}(G)$ is true for every graph [14], while the equality holds for bipartite graphs [15], and for unicyclic non-König-Egerváry graphs [16].

In this paper, we prove that if G is an almost bipartite non-König-Egerváry graph, then:
(i) $\text{core}(G)=\ker(G)$, like for bipartite graphs;
(ii) $\text{corona}(G)\cup N(\text{core}(G))=V$, like for König-Egerváry graphs.

1 Introduction

Throughout this paper $G=(V,E)$ is a finite, undirected, loopless graph without multiple edges, with vertex set $V=V(G)$ of cardinality $n(G)$, and edge set $E=E(G)$ of size $m(G)$.

If $X\subset V$, then $G[X]$ is the subgraph of G spanned by X. By $G-W$ we mean the subgraph $G[V-W]$, if $W\subset V(G)$. For $F\subset E(G)$, by $G-F$ we denote the subgraph of G obtained by deleting the edges of F, and we use $G-e$, if $F=\{e\}$. If $A,B\subset V$ and $A\cap B=\emptyset$, then (A,B) stands for the set $\{e=ab: a\in A, b\in B, e\in E\}$.

The neighborhood of a vertex $v\in V$ is the set $N(v)=\{w: w\in V \textit{ and } vw\in E\}$, and $N(A)=\bigcup\{N(v): v\in A\}$, $N[A]=A\cup N(A)$ for $A\subset V$. By C_n, K_n we mean the chordless cycle on $n\geq 3$ vertices, and respectively the complete graph on $n\geq 1$ vertices. In order to avoid ambiguity, we use also $N_G(v)$ instead of $N(v)$, and $N_G(A)$ instead of $N(A)$.

A *cycle* is a trail, where the only repeated vertices are the first and last ones. The graph G is *unicyclic* if it has a unique cycle.

Let us define the *trace* of a family $\mathcal{F}$ of sets on the set X as $\mathcal{F}|_X=\{F\cap X: F\in\mathcal{F}\}$.

A set S of vertices is *independent* if no two vertices from S are adjacent, and an independent set of maximum size will be referred to as a *maximum independent set*. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of a maximum independent set of G. By $\text{Ind}(G)$ we mean the family of all independent sets of G.

[1]Email: levitv@ariel.ac.il
[2]Email: eugen_m@hit.ac.il

Let $\Omega(G) = \{S : S \textit{ is a maximum independent set of } G\}$, $\text{core}(G) = \bigcap\{S : S \in \Omega(G)\}$ [9], and $\text{corona}(G) = \bigcup\{S : S \in \Omega(G)\}$ [1].

An edge $e \in E(G)$ is *α-critical* whenever $\alpha(G-e) > \alpha(G)$. Notice that $\alpha(G) \leq \alpha(G-e) \leq \alpha(G)+1$ holds for each edge e.

The number $d(G) = \max\{|X| - |N(X)| : X \subseteq V\}$ is called the *critical difference* of G, and a set $A \subseteq V(G)$ is *critical* if $|A| - |N(A)| = d(G)$ [19]. The number $id(G) = \max\{d_G(I) : I \in \text{Ind}(G)\}$ is called the *critical independence difference* of G. If $A \subseteq V(G)$ is independent and $d_G(A) = id(G)$, then A is a *critical independent set* [19].

Theorem 1. *[19] The equality $d(G) = id(G)$ holds for every graph G.*

If $\alpha(G) + \mu(G) = n(G)$, then G is called a *König-Egerváry graph* [4, 18]. Various properties of König-Egerváry graphs are presented in [2, 3, 6, 8, 10, 11, 12]. It is known that every bipartite graph is a König-Egerváry graph [5, 7]. This class includes also non-bipartite graphs (see, for instance, the graph G in Figure 1).

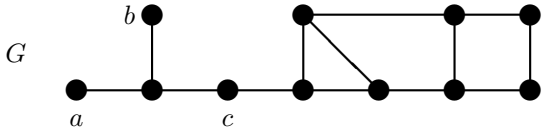


Figure 1: G is a König-Egerváry graph with $\text{core}(G) = \{a, b, c\}$ and $\ker(G) = \{a, b\}$.

For a graph G, let $\ker(G) = \bigcap\{S : S \textit{ is a critical independent set}\}$.

Theorem 2. (i) *[14]* $\ker(G)$ *is the unique minimal critical (independent) set of G, and* $\ker(G) \subseteq \text{core}(G)$ *is true for every graph.*

(ii) *[15, 16] If G is a bipartite or a unicyclic non-König-Egerváry graph, then* $\ker(G) = \text{core}(G)$.

(iii) [10, 16] $\text{corona}(G) \cup N(\text{core}(G) = V(G)$ *holds for every König-Egerváry or unicyclic graph G.*

A graph G is *almost bipartite* if it has a unique odd cycle, denoted $C = (V(C), E(C))$ [17]. For every $y \in V(C)$, let $D_y = (V_y, E_y)$ be the connected induced subgraph of $G - E(C)$ containing y. Clearly, every unicyclic graph with an odd cycle is almost bipartite.

Theorem 3. *[17] If G is an almost bipartite non-König-Egerváry graph, then*

$$\Omega(G)|_{V(D_y - y)} = \Omega(D_y - y)$$

for every $y \in V(C)$.

An almost bipartite König-Egerváry graph may have $\ker(G) \neq \text{core}(G)$; e.g., the graphs in Figure 2 have $\ker(G_1) = \emptyset$ and $\ker(G_1) = \{u, v\}$.

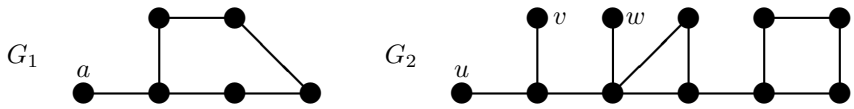


Figure 2: G_1, G_2 are König-Egerváry graphs with $\text{core}(G_1) = \{a\}$, $\text{core}(G_2) = \{u, v, w\}$.

If $H_j, j = 1, 2, ..., k$, are all the connected components of G, it is easy to see that

$$\Omega(G) = \bigcup_{j=1}^{k} \Omega(H_j), \ \text{core}(G) = \bigcup_{j=1}^{k} \text{core}(H_j) \ \text{ and } \ker(G) = \bigcup_{j=1}^{k} \ker(H_j).$$

In this paper, we show that $\ker(G) = \text{core}(G)$ and $\text{corona}(G) \cup N(\text{core}(G)) = V(G)$ hold for every almost bipartite non-König-Egerváry graph G. Since both equalities hold for every bipartite connected component of G, we may assume that the almost bipartite non-König-Egerváry graphs are connected.

2 Results

Proposition 4. *If G is almost bipartite with $C = (V(C), E(C))$ as its unique odd cycle, then $V(D_a) \cap V(D_b) = \emptyset$ for every two different vertices $a, b \in V(C)$.*

Proof. Assume, to the contrary, that there exist $a, b \in V(C)$, such that $V(D_a) \cap V(D_b) \neq \emptyset$. Let $x \in V(D_a) \cap V(D_b)$. Thus, there exists some path containing x, and connecting a and b. Let P_1 be a shortest one of this kind. On the other hand, there exist two paths, say P_2 and P_3, connecting a and b, and containing only vertices belonging to C. Therefore, either P_1 and P_2, or P_1 and P_3, give birth to an odd cycle, different from C, and thus contradicting the fact that C is the unique odd cycle of G. □

Lemma 5. *[17] If G is an almost bipartite graph, then the following assertions hold:*
(i) $n(G) - 1 \leq \alpha(G) + \mu(G) \leq n(G)$;
(ii) $n(G) - 1 = \alpha(G) + \mu(G)$ *if and only if each edge of its unique odd cycle is α-critical.*

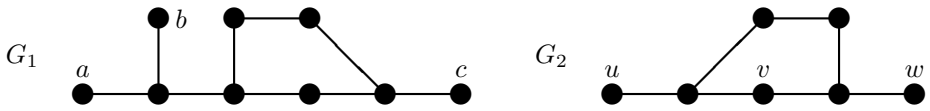


Figure 3: G_1, G_2 are König-Egerváry graphs, $\text{core}(G_1) = \{a, b, c\}$, $\text{core}(G_2) = \{u, v, w\}$.

Lemma 6. *[17] Let G be an almost bipartite non-König-Egerváry graph with the unique odd cycle C.*
(i) *If A is a critical independent set, then $A \cap V(C) = \emptyset$.*
(ii) $\text{core}(G)$ *is critical.*

Lemma 7. *[13] For every bipartite graph H, a vertex $v \in \text{core}(H)$ if and only if there exists a maximum matching that does not saturate v.*

Lemma 8. *Let G be an almost bipartite graph. If there is $x \in N_1(C)$, such that $x \in \text{core}(D_y - y)$ for some $y \in V(C)$, then G is a König-Egerváry graph.*

Proof. Let $x \in \text{core}(D_y - y)$, $y \in N(x) \cap V(C)$, and $z \in N(y) \cap V(C)$. Suppose, to the contrary, that G is not a König-Egerváry graph. By Lemma 5*(i)* and *(ii)*, the edge yz is α-critical. According to Lemma 6, we obtain that $y \notin \text{core}(G)$. Thus it follows that $\alpha(G) = \alpha(G - y)$. By Lemma 7 there exists a maximum matching M_x of $D_y - y$ not saturating x. Combining M_x with a maximum matching of $G - D_y$ we get a maximum matching M_y of $G - y$. Hence $M_y \cup \{xy\}$ is a matching of G, which results in $\mu(G) \geq \mu(G - y) + 1$. Consequently, using Lemma 5*(ii)* and having in mind that $G - y$ is a bipartite graph of order $n(G) - 1$, we get the following contradiction

$$n(G) - 1 = \alpha(G) + \mu(G) \geq \alpha(G - y) + \mu(G - y) + 1 = n(G) - 1 + 1 = n(G),$$

that completes the proof. □

Theorem 9. *[17] If G is an almost bipartite non-König-Egerváry graph, then*
(i) *core*$(G) \cap N[V(C)] = \emptyset$;
(ii) $\text{core}(G) = \bigcup_{y \in V(C)} \text{core}(D_y - y)$.

Theorem 10. *Let G be an almost bipartite non-König-Egerváry graph with the unique odd cycle C. Then*

$$\ker(G) = \bigcup_{y \in V(C)} \ker(D_y - y) = \bigcup_{y \in V(C)} \text{core}(D_y - y) = \text{core}(G).$$

Proof. By Theorem 9*(ii)*, we have that $\operatorname{core}(G) = \bigcup_{y \in V(C)} \operatorname{core}(D_y - y)$.

Since every $D_y - y$ is a bipartite graph, we infer that $\ker(D_y - y) = \operatorname{core}(D_y - y)$, by Theorem 2*(ii)*. Consequently, we infer that

$$\operatorname{core}(G) = \bigcup_{y \in V(C)} \operatorname{core}(D_y - y) = \bigcup_{y \in V(C)} \ker(D_y - y).$$

According to Lemma 6*(ii)*, the set $\operatorname{core}(G)$ is critical in G. Hence, we get that

$$\ker(G) \subseteq \bigcup_{y \in V(C)} \ker(D_y - y).$$

Thus it is enough to show that $\bigcup_{y \in V(C)} \ker(D_y - y) \subseteq \ker(G)$. In other words,

$$\ker(D_y - y) \subseteq \ker(G)|_{V(D_y - y)}$$

for every $y \in V(C)$, which is equivalent to the fact that $\ker(G)|_{V(D_y-y)}$ is critical in $D_y - y$.

By Lemma 8, if $A \subseteq \operatorname{core}(D_y - y)$, then $N_G(A) = N_{D_y - y}(A)$, since G is a non-Konig-Egervary almost bipartite graph. Hence, $d_G(A) = d_{D_y-y}(A)$ for every $A \subseteq \ker(D_y - y)$. Thus in accordance with Theorem 2*(i)*, if $A \subset \ker(D_y - y)$, then

$$d_G(A) = d_{D_y-y}(A) < d_{D_y-y}(\ker(D_y - y)) = d_G(\ker(D_y - y)). \qquad (*)$$

Since $\ker(G) \subseteq \bigcup_{y \in V(C)} \ker(D_y - y)$,

$$d_G(\ker(G)) = d_G\left(\ker(G) \cap \bigcup_{y \in V(C)} \ker(D_y - y)\right) =$$

$$d_G\left(\bigcup_{y \in V(C)} (\ker(D_y - y) \cap \ker(G))\right) = \sum_{y \in V(C)} d_G(\ker(D_y - y) \cap \ker(G)).$$

If $\ker(G) \neq \bigcup_{y \in V(C)} \ker(D_y - y)$, then

$$\ker(G)|_{V(D_y-y)} \subset \ker(D_y - y)$$

for some $y \in V(C)$. Consequently, using the inequality (*) for $A = \ker(G)|_{V(D_y-y)}$, we obtain

$$d_G(\ker(G)) = \sum_{y \in V(C)} d_G\left(\ker(G)|_{V(D_y-y)}\right) <$$

$$\sum_{y \in V(C)} d_G(\ker(D_y - y)) = d_G\left(\bigcup_{y \in V(C)} \ker(D_y - y)\right) = d(\operatorname{core}(G)) = d(G),$$

which stays in contradiction with the fact that $\ker(G)$ is critical in G (by Theorem 2*(i)*). □

Proposition 11. *[9] For a connected bipartite graph $G = (A, B, E)$ of order at least two, the following assertions are true:*

(i) $\alpha(G) > |V(G)|/2$ *if and only if* $|\operatorname{core}(G)| \geq 2$*;*

(ii) $\alpha(G) = |V(G)|/2$ *if and only if* $|\operatorname{core}(G)| = 0$ *and* $A, B \in \Omega(G)$.

Corollary 12. *If G is an almost bipartite non-König-Egerváry graph, then* $|\text{core}(G)| \neq 1$.

Proof. By Theorem 10 we have that $\bigcup_{y \in V(C)} \text{core}\,(D_y - y) = \text{core}\,(G)$, while by Proposition 11, we know that $|\text{core}(D_y - y)| \neq 1$ for every $y \in V(C)$, since $D_y - y$ is bipartite. Hence $|\text{core}(G)| \neq 1$. □

It is worth noticing that there exists an almost bipartite König-Egerváry graph with a critical independent set meeting its unique cycle; e.g., the bull graph.

As we already know from Theorem 2*(iii)*, the equality $\text{corona}(G) \cup N(\text{core}(G)) = V(G)$ holds for every König-Egerváry or unicyclic graph. For other graphs, this equality is not mandatory. For instance, consider the graphs in Figure 4:

$$\text{corona}(G_1) \cup N(\text{core}(G_1)) = V(G_1), \text{ while } \text{corona}(G_2) \cup N(\text{core}(G_2)) = V(G_2) - \{v\}.$$

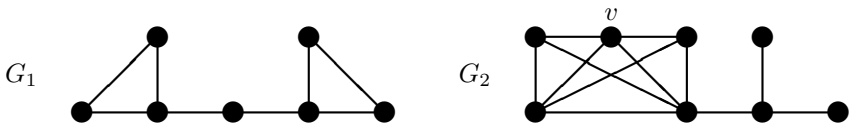


Figure 4: G_1, G_2 are non-König-Egerváry graphs.

Theorem 13. *If G is an almost bipartite non-König-Egerváry graph, then*

(i) $\text{corona}(G) \cup N(\text{core}(G)) = V(G)$*;*

(ii) $\text{corona}(G) = V(C) \cup \left(\bigcup_{y \in V(C)} \text{corona}(D_y - y) \right)$.

Proof. (i) It is enough to show that $V(G) \subseteq \text{corona}(G) \cup N(\text{core}(G))$.

Let $a \in V(G)$.

Case 1. $a \in V(C)$. If $b \in N(a) \cap V(C)$, then, by Lemma 5, the edge ab is α-critical. Hence $a \in \text{corona}(G)$.

Case 2. $a \in V(G) - V(C)$. It follows that $a \in V(D_y - y)$, for some $y \in V(C)$.

Since $G[D_y - y]$ is bipartite, by Theorem 2*(iii)*, we know that $V(D_y - y) = \text{corona}(D_y - y) \cup N(\text{core}(D_y - y))$, while by Theorem 3, we have that $\Omega(G)|_{V(D_y - y)} = \Omega(D_y - y)$ for every $y \in V(C)$, which ensures that $\text{corona}(D_y - y) \subseteq \text{corona}(G)$.

Therefore, either $a \in \text{corona}(D_y - y) \subseteq \text{corona}(G)$, or $a \in N(\text{core}(D_y - y)) \subseteq N(\text{core}(G))$, because $\text{core}\,(D_y - y) \subseteq \text{core}\,(G)$, by Theorem 9*(ii)*. Thus, $a \in \text{corona}(G) \cup N(\text{core}(G))$.

All in all, $V(G) = \text{corona}(G) \cup N(\text{core}(G))$.

(ii) In the proof of Part *(i)* we showed that $\text{corona}(D_y - y) \subseteq \text{corona}(G)$ for every $y \in V(C)$, and $V(C) \subseteq \text{corona}(G)$.

Hence, $V(C) \cup \left(\bigcup_{y \in V(C)} \text{corona}(D_y - y) \right) \subseteq \text{corona}(G)$. To complete the proof, it remains to validate that $\text{corona}(G) \subseteq V(C) \cup \left(\bigcup_{y \in V(C)} \text{corona}(D_y - y) \right)$. Let $a \in \text{corona}(G)$. Then, $a \in S$ for some $S \in \Omega(G)$. Suppose $a \notin V(C)$, then there must be $y \in V(C)$ such that $a \in D_y - y$. Thus, $a \in S \cap V(D_y - y) \subseteq \text{corona}(D_y - y)$, because $\Omega(G)|_{V(D_y - y)} = \Omega(D_y - y)$, in accordance with Theorem 3. □

3 Conclusions

It is known that for every graph $\ker(G) \subseteq \operatorname{core}(G)$. We know that $\ker(G) = \operatorname{core}(G)$ holds for both bipartite and almost bipartite non-König-Egerváry graphs. These findings motivate the following.

Problem 14. *Characterize graphs enjoying* $\ker(G) = \operatorname{core}(G)$.

References

[1] E. Boros, M. C. Golumbic, and V. E. Levit. "On the number of vertices belonging to all maximum stable sets of a graph". In: *Discrete Appl. Math.* 124.1-3 (2002), pp. 17–25. DOI: 10.1016/S0166-218X(01)00327-4.

[2] J.-M. Bourjolly, P. L. Hammer, and B. Simeone. "Node-weighted graphs having the König-Egerváry property". In: *Math. Programming Stud.* 22 (1984), pp. 44–63. DOI: 10.1007/bfb0121007.

[3] J.-M. Bourjolly and W. R. Pulleyblank. "König-Egerváry graphs, 2-bicritical graphs and fractional matchings". In: *Discrete Appl. Math.* 24.1-3 (1989), pp. 63–82. DOI: 10.1016/0166-218X(92)90273-D.

[4] R. W. Deming. "Independence numbers of graphs—an extension of the Koenig-Egervary theorem". In: *Discrete Math.* 27.1 (1979), pp. 23–33. DOI: 10.1016/0012-365X(79)90066-9.

[5] E. Egerváry. "On combinatorial properties of matrices". In: *Matematikai Lapok* 38 (1931), pp. 16–28.

[6] A. Jarden, V. E. Levit, and E. Mandrescu. "Two more characterizations of König-Egerváry graphs". In: *Discrete Appl. Math.* 231 (2017), pp. 175–180. DOI: 10.1016/j.dam.2016.05.012.

[7] D. König. "Graphen und Matrizen". In: *Matematikai Lapok* 38 (1931), pp. 116–119.

[8] C. E. Larson. "The critical independence number and an independence decomposition". In: *European J. Combin.* 32.2 (2011), pp. 294–300. DOI: 10.1016/j.ejc.2010.10.004.

[9] V. E. Levit and E. Mandrescu. "Combinatorial properties of the family of maximum stable sets of a graph". In: *Discrete Appl. Math.* 117.1-3 (2002), pp. 149–161. DOI: 10.1016/S0166-218X(01)00183-4.

[10] V. E. Levit and E. Mandrescu. "On α^+-stable König-Egerváry graphs". In: *Discrete Math.* 263.1-3 (2003), pp. 179–190. DOI: 10.1016/S0012-365X(02)00528-9.

[11] V. E. Levit and E. Mandrescu. "On α-critical edges in König-Egerváry graphs". In: *Discrete Math.* 306.15 (2006), pp. 1684–1693. DOI: 10.1016/j.disc.2006.05.001.

[12] V. E. Levit and E. Mandrescu. "Critical independent sets and König-Egerváry graphs". In: *Graphs Combin.* 28.2 (2012), pp. 243–250. DOI: 10.1007/s00373-011-1037-y.

[13] V. E. Levit and E. Mandrescu. "On the core of a unicyclic graph". In: *Ars Math. Contemp.* 5.2 (2012), pp. 321–327. DOI: 10.26493/1855-3974.201.6e1.

[14] V. E. Levit and E. Mandrescu. "Vertices belonging to all critical sets of a graph". In: *SIAM J. Discrete Math.* 26.1 (2012), pp. 399–403. DOI: 10.1137/110823560.

[15] V. E. Levit and E. Mandrescu. "Critical sets in bipartite graphs". In: *Ann. Comb.* 17.3 (2013), pp. 543–548. DOI: 10.1007/s00026-013-0195-4.

[16] V. E. Levit and E. Mandrescu. "On the intersection of all critical sets of a unicyclic graph". In: *Discrete Appl. Math.* 162 (2014), pp. 409–414. DOI: 10.1016/j.dam.2013.09.006.

[17] V. E. Levit and E. Mandrescu. "On the critical difference of almost bipartite graphs". In: *Journal of Algebraic Combinatorics* (2020). DOI: 10.1007/s10801-020-00968-x.

[18] F. Sterboul. “A characterization of the graphs in which the transversal number equals the matching number”. In: *J. Combin. Theory Ser. B* 27.2 (1979), pp. 228–229. DOI: 10.1016/0095-8956(79)90085-6.

[19] C. Q. Zhang. “Finding critical independent sets and critical vertex subsets are polynomial problems”. In: *SIAM J. Discrete Math.* 3.3 (1990), pp. 431–438. DOI: 10.1137/0403037.

On local bipartite Moore graphs

Nacho López[1*], Josep Conde[2*], Gabriela Araujo-Pardo[3†]

[*]Departament de Matemàtica, Universitat de Lleida, Jaume II 69, 25001 Lleida, Spain.
[†]Instituto de Matemáticas, Universidad Nacional Autonóma de México, México D.F. 04510, México.

Abstract

Given the values of the maximum degree d and the (even) girth g of a bipartite graph, there is a natural lower bound for its number of vertices known as the *bipartite Moore bound*. Graphs attaining such a bound are referred to as *bipartite Moore graphs*. The fact that there are very few bipartite Moore graphs suggests the definition of local bipartite Moore graphs. Here we consider the problem of classifying these graphs according to their 'proximity' to some properties that a theoretical bipartite Moore graph should have.

1 Introduction

The modelling of interconnection networks by graphs motivated the study of several extremal problems that involve three well known parameters of a graph (degree, girth and order) and ask for the optimal value of one of them while holding the other two fixed. For instance, the *degree/girth problem* (also known as *the cage problem*) consists in finding the smallest order of a graph with prescribed degree and girth (for a survey of it see [4]). We are also interested in some particular versions of the problem, namely when the graphs are restricted to a certain class, such as the class of bipartite graphs. In this case, given the values of the maximum degree d and the (even) girth $g = 2k$ of a graph, there is a natural lower bound for its number of vertices n,

$$n \geq M(d,g) = 2(1 + (d-1) + (d-1)^2 + \cdots + (d-1)^{g/2-1}) = 2\frac{(d-1)^{g/2} - 1}{d-2} \tag{1}$$

where $M(d,g)$ is known as the *biparite Moore bound*. Graphs attaining such a bound are referred to as *biparite Moore graphs*. For $d = 2$, cycles of length g are the only bipartite Moore graphs; while, for $d \geq 3$, their existence is possible only for $g = 4, 6, 8$ or 12 (see Feit and Higman [5], Singleton [7]). In the particular case $g = 4$ and $d \geq 3$, bipartite Moore graphs are the complete bipartite graphs, but for the remaining values of g, bipartite Moore graphs of degree d have been constructed only when $d-1$ is a prime power [6]. In general, the question of whether or not bipartite Moore graphs of girth $6, 8$ or 12 for other values of d exist remains open.

The fact that there are very few biparite Moore graphs suggested the study of graphs 'close' to the Moore ones. This 'closeness' has usually been measured as the difference between the (unattainable) Moore bound and the order of the considered graphs. In this sense, the existence and construction of graphs with small 'defect' δ has deserved much attention in the literature (see [6]). Another kind of approach considers relaxing some of the constraints implied by the Moore bound. For instance, this approach has already been considered in the context of the degree/diameter problem, where Tang, Miller

[1]Email: nacho.lopez@udl.cat. Research of N. López is supported in part by the MCIN/AEI/10.13039/501100011033 through Grant PID2020-115442RB-I00 (Spanish Ministerio de Ciencia e Innovacion)

[2]Email:josep.conde@udl.cat

[3]Email: garaujo@matem.unam.mx. Research of G. Araujo was supported by PASPA-DGAPA and CONACyT Sabbatical Stay 2020, CONACyT-México under Project 282280, 47510664 and PAPIIT-México under Projects IN107218, IN106318.

and Lin [8] relax the condition of the degree and admit few vertices with degree $d+\epsilon$. Alternatively, Capdevila et al. [2] allow the existence of vertices with eccentricity $k+1$. In this context, regular graphs of degree d, radius k, diameter $k+1$ and order equal to the Moore bound are known as *radial Moore graphs* (see [3]).

Terminology and notation

We follow the notation given by Capdevila et al. in [2]. Let $G=(V,E)$ be a connected graph. Given two vertices u and v of G, the distance between u and v, $d(u,v)$, is the length of a shortest path joining them. The sum of all distances to a vertex v, $s(v)=\sum_{u\in V} d(u,v)$, is referred to as the *status* of v. We define the *status vector* of G, $\mathbf{s}(G)$, as the vector constituted by the status of all its vertices. Usually, when the vector is long enough, we denote it with a short description using superscripts, that is, $\mathbf{s}(G): s_1^{n_1}, s_2^{n_2}, \ldots, s_k^{n_k}$, where $s_1 > s_2 > \cdots > s_k$, and n_i denotes the number of vertices having s_i as its local status, for all $1 \leq i \leq k$. The *total status* of G, $s(G)$, is the sum of all its statuses:

$$s(G) = \|\mathbf{s}(G)\|_1 = \sum_{v\in V} s(v) = \sum_{u,v\in V} d(u,v).$$

This invariant is twice the *Wiener index* of G.

We recall that the *girth* of a graph G is the length of its shortest cycle. If we restrict our attention to the cycles through a given vertex v, we can define the *girth* of v, $g(v)$, as the smallest length of such 'rooted' cycles. The vector $\mathbf{g}(G)$ constituted by the girths of all its vertices will be referred to as the *girth vector* of G, that is, $\mathbf{g}(G): g_1^{n_1}, g_2^{n_2}, \ldots, g_k^{n_k}$, where $g_1 > g_2 > \cdots > g_k$, and n_i denotes the number of vertices having g_i as its local girth, for all $1 \leq i \leq k$.

2 Local bipartite Moore graphs and ranking measures

Bipartite Moore graphs can be seen as extremal regular graphs in two different ways, since their order attains the maximum [respectively, minimum] value allowed by their degree d and diameter k [respectively, girth $g=2k$], according to the bipartite Moore bound (1). Intuitively, if we hang a bipartite Moore graph from any of its edges uv, we observe the 'same' distance-preserving spanning tree (see Figure 1). Notice that there are $(d-1)$ vertices 'hanging' from u (and also from v) in this spanning tree. Each of these vertices at distance 1 from u have $d-1$ vertices at distance 2 and so on, generating a spanning tree as in Figure 1. Taking into account that the girth of the graph cannot be less than $g=2k$, we obtain that this graph must have at least $M(d,g)$ vertices.

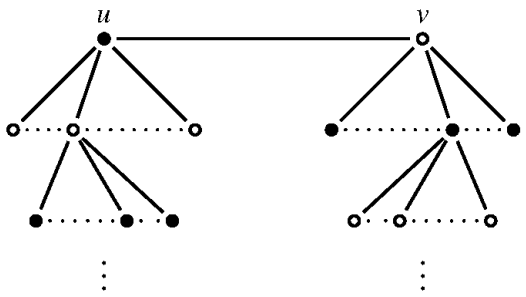


Figure 1: Distance-preserving spanning tree of a bipartite Moore graph.

In [1] local bipartite Moore graphs are introduced: given two positive integers $d \geq 2$ and $g=2k \geq 6$, a connected regular bipartite graph of degree d and order $n=M(d,g)$ is said to be a local bipartite Moore graph if it contains at least one edge uv such that its corresponding distance-preserving spanning tree is as in Fig. 1. From this point of view, bipartite Moore graphs are a particular case of local bipartite Moore graphs having the same distance-preserving spanning tree for any of its edges. There

are at least two vertices with local girth g in a local bipartite Moore graph (the end vertices of the edge uv in Fig. 1); while, for a bipartite Moore graph, all vertices have local girth g (and hence the whole graph has girth g).

Proposition 1. *Given two positive integers $d > 2$ and $g = 2k > 4$, let G be a local bipartite Moore graph of degree d and order $n = M(d, g)$. Then, for every vertex v of G we have*

$$s(v) \geq \frac{\left(k(d-2)^2 + (k-1)d(d-2) - d\right)(d-1)^{k-1} + d}{(d-2)^2} \quad \text{and} \quad g(v) \leq 2k.$$

Moreover, these bounds are attained for every vertex if and only if G is a bipartite Moore graph.

Let us denote by $\mathcal{RB}(d, g)$ the set of all nonisomorphic connected regular bipartite graphs of degree d and order $M(d, g)$. The set of local bipartite Moore graphs of degree d and girth g is denoted as $\mathcal{LBM}(d, g)$. Of course $\mathcal{LBM}(d, g) \subseteq \mathcal{RB}(d, g)$. Our main goal is to obtain all the graphs in $\mathcal{LBM}(d, g)$, for some values of (d, g), and to rank all of them according to their closeseness to a bipartite Moore graph. To this end, let $\mathbf{g}_{d,g}$ be the n-vector with all components equal to g. Notice that $\mathbf{g}_{d,g}$ represents the girth vector of a bipartite Moore graph of degree d and girth g. For every positive integer p we define

$$\tilde{N}_p(G) = \|\mathbf{g}(G) - \mathbf{g}_{d,g}\|_p.$$

In particular,

$$\tilde{N}_1(G) = \sum_{v \in V} (g - g(v)).$$

Given two graphs $G_1, G_2 \in \mathcal{RB}(d, g)$ we define G_1 and G_2 to be *girth-equivalent*, $G_1 \approx G_2$, if they have the same girth vector. In the quotient set of $\mathcal{LBM}(d, g)$ by $\approx$, $\mathcal{LBM}(d, g)/\approx$, we say that G_1 is closer than G_2 to being a bipartite Moore graph if there exists a positive integer l such that

$$\tilde{N}_p(G_1) = \tilde{N}_p(G_2),\ p = 1, \ldots, l-1 \quad \text{and} \quad \tilde{N}_l(G_1) < \tilde{N}_l(G_2),$$

in which case we denote $G_1 \prec G_2$. Besides, let $\mathbf{s}_{d,k}$ be the vector of dimension $n = M(d, g)$ whose components are all equal to the status value that a bipartite Moore graph should have. Again, we define

$$N_p(G) = \|\mathbf{s}(G) - \mathbf{s}_{d,k}\|_p.$$

In particular, $N_1(G)$ measures the difference between the total status of G and that corresponding to a bipartite Moore graph.

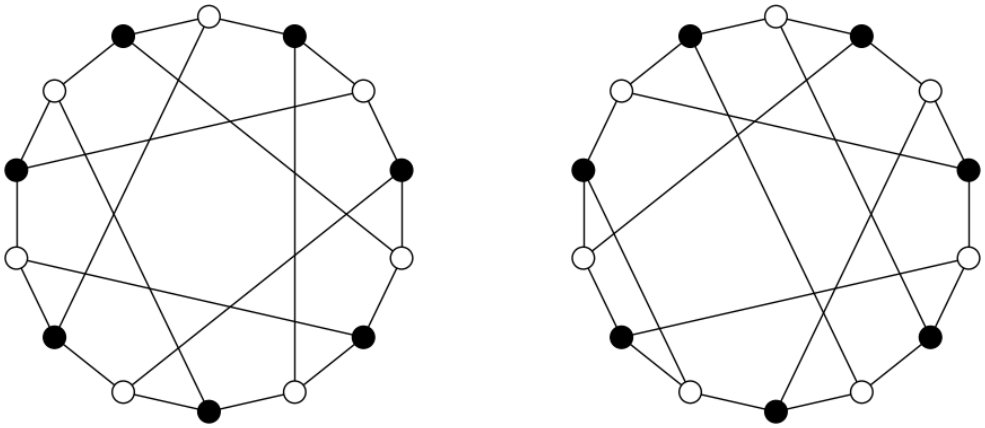
Given two graphs $G_1, G_2 \in \mathcal{LBM}(d, g)$, we define G_1 and G_2 to be *status-equivalent*, $G_1 \sim G_2$, if they have the same status vector. In the quotient set of $\mathcal{LBM}(d, g)$ by $\sim$, $\mathcal{LBM}(d, g)/\sim$, we say that G_1 is closer than G_2 to being a bipartite Moore graph if there exists a positive integer l such that

$$N_p(G_1) = N_p(G_2),\ p = 1, \ldots, l-1 \quad \text{and} \quad N_l(G_1) < N_l(G_2),$$

in which case we denote $G_1 < G_2$.

3 Ranking of cubic local bipartite Moore graphs for $g = 6$ and $g = 8$.

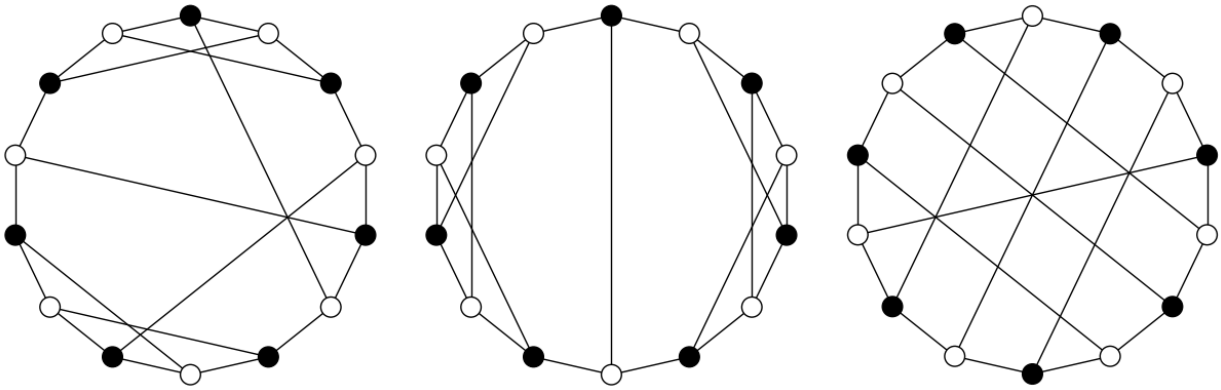
The set $\mathcal{LBM}(3, 6)$ contains exactly 5 graphs (see [1]). They are shown in Figures 2 and 3. As a consequence, $\tilde{N}_1(G_1) = N_1(G_1) = 16$ and G_1 is precisely the closest graph to the unique bipartite Moore graph for $d = k = 3$ (the Heawood graph). Moreover, $\tilde{N}_1(G_2) = 20 < \tilde{N}_1(G_3) = \tilde{N}_1(G_4) = 24$ and $N_1(G_3) = 32$, $N_1(G_2) = 40$ and $N_1(G_4) = 96$. The rank order is therefore $H \prec G_1 \prec G_2 \prec G_3 \approx G_4$ for the girth norm, while it is $H < G_1 < G_3 < G_2 < G_4$ for the status norm.



H
$\mathbf{g}(H) : 6^{14}$,
$\mathbf{s}(H) : 27^{14}$

G_1
$\mathbf{g}(G_1) : 6^6, 4^8$,
$\mathbf{s}(G_1) : 27^6, 29^8$

Figure 2: The Heawood graph H (unique bipartite Moore graph for $d = k = 3$) and its closest graph G_1, together with their corresponding statuses and girth vectors.



G_2
$\mathbf{g}(G_2) : 6^4,\ 4^{10}$
$\mathbf{s}(G_2) : 27^4, 31^{10}$

G_3
$\mathbf{g}(G_3) : 6^2,\ 4^{12}$
$\mathbf{s}(G_3) : 27^2, 29^8, 31^4$

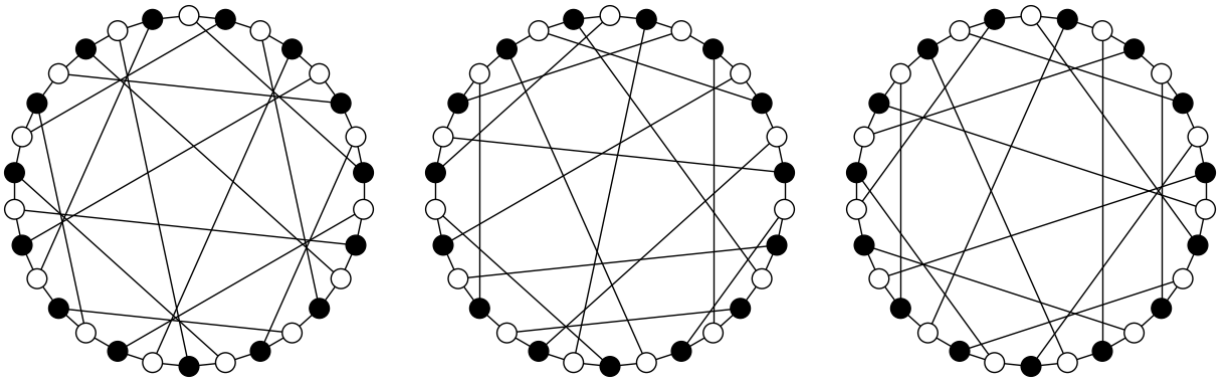
G_4
$\mathbf{g}(G_4) : 6^2,\ 4^{12}$
$\mathbf{s}(G_4) : 27^2, 31^4, 37^8$

Figure 3: Remaining graphs in $\mathcal{LBM}(3, 6)$ (other than H and G_1) and their corresponding statuses and girth vectors.

Theorem 2. *According to both status norm and girth norm, G_1 is the closest graph to the unique bipartite Moore graph for $d = 3$ and $g = 6$.*

There are 6951 graphs in $\mathcal{LBM}(3, 8)$, which represents only around 0.03% of the total graphs in $\mathcal{RB}(3, 8)$. Figure 4 shows the Tutte-Coxeter graph and the closest graphs H_1 and H_2 according to the status-norm and girth-norm, respectively. In this particular case, $\tilde{N}_1(H_1) = 36$ and $\tilde{N}_1(H_2) = 48$. Hence $H_1 \prec H_2$. Besides $N_1(H_1) = N_1(H_2) = 48$, but $N_2(H_1) > N_2(H_2)$. As a consequence $H_2 < H_1$.

Theorem 3. *H_1 is the closest graph to the unique bipartite Moore graph for $d = 3$ and $g = 8$ according to the girth-norm, while H_2 is the closest graph according to the status-norm.*



T
$\mathbf{g}(T) : 8^{30}$
$\mathbf{s}(T) : 83^{30}$

H_1
$\mathbf{g}(H_1) : 8^{12}, 6^{18}$
$\mathbf{s}(H_1) : 83^{12}, 85^{12}, 87^{6}$

H_2
$\mathbf{g}(H_2) : 8^{6}, 6^{24}$
$\mathbf{s}(H_2) : 83^{6}, 85^{24}$

Figure 4: First graphs in the ranking of $\mathcal{LBM}(3, 8)$. The Tutte-Coxeter graph is the unique bipartite Moore graph in this case and it has been denoted by T.

References

[1] G. Araujo-Pardo and N. López. *On new record graphs close to bipartite Moore graphs*. accepted in Graphs and Combinatorics. 2022. DOI: 10.48550/ARXIV.2005.02427.

[2] C. Capdevila, J. Conde, G. Exoo, J. Gimbert, and N. López. "Ranking measures for radially Moore graphs". In: *Networks* 56.4 (2010), pp. 255–262. DOI: 10.1002/net.20377.

[3] G. Exoo, J. Gimbert, N. López, and J. Gómez. "Radial Moore graphs of radius three". In: *Discrete Appl. Math.* 160.10-11 (2012), pp. 1507–1512. DOI: 10.1016/j.dam.2012.02.023.

[4] G. Exoo and R. Jajcay. "Dynamic cage survey". In: *Electron. J. Combin.* DS16.Dynamic Surveys (2008), p. 48. DOI: 10.37236/37.

[5] W. Feit and G. Higman. "The nonexistence of certain generalized polygons". In: *J. Algebra* 1 (1964), pp. 114–131. DOI: 10.1016/0021-8693(64)90028-6.

[6] M. Miller and J. Širáň. "Moore graphs and beyond: a survey of the degree/diameter problem". In: *Electron. J. Combin.* DS14.Dynamic Surveys (2005), p. 61. DOI: 10.37236/35.

[7] R. Singleton. "On minimal graphs of maximum even girth". In: *J. Combinatorial Theory* 1 (1966), pp. 306–332.

[8] J. Tang, M. Miller, and Y. Lin. "HSAGA and its application for the construction of near-Moore digraphs". In: *J. Discrete Algorithms* 6.1 (2008), pp. 73–84. DOI: 10.1016/j.jda.2007.05.003.

Sequences related to square and cube zig-zag shapes

László Németh[1], László Szalay[2]

Inst. of Informatics and Mathematics, University of Sopron, Hungary

The full version of this work partly can be found in [8] and partly will be published elsewhere.

Abstract

Considering a so-called square k-zig-zag shape as a part of the regular square grid as a k-zig-zag digraph, we give values to its vertices according to the number of the shortest paths from a base vertex. It provides several integer sequences, whose higher-order homogeneous recurrences are determined by the help of a special matrix recurrence. We also define a special zig-zag shape based on the spatial cube grid, and we give recurrence relation for one of their digraph walk.

1 Introduction

The present extended abstract summarizes the studies of diagonal and zig-zag paths on a particular $k+1$ wide, infinite part of the usual square lattice, and along these paths we determine linear recurrence sequences that are mostly defined in the *On-Line Encyclopedia of Integer Sequences* (OEIS, [9]) without combinatorial interpretations. In this manner our investigation, among others, gives them geometrical and combinatorial background. The consideration of zig-zag shapes is not an isolated challenge. For example, Baryshnikov and Romik [2] examined the so-called Young diagrams, which are similar to our construction, and defined a kind of 'zig-zag' numbers by the help of the alternating permutations. Stanley [10] published a survey in which he dealt with the 'zig-zag' shapes and the alternating permutations. Recently, Ahmad et al. [1] studied some graph-theoretic properties of special zig-zag polyomino chains.

The authors proved in [3, 6] that all the integer linear homogeneous recurrence sequences $\{f_i\}_{i\geq 0}$ defined by

$$f_i = \alpha f_{i-1} \pm f_{i-2}, \qquad (i \geq 2),$$

where $\alpha \in \mathbb{N}$, $\alpha \geq 2$, and $f_0 < f_1$ are positive integers with $\gcd(f_0, f_1) = 1$, appear along corresponding zig-zag paths in the hyperbolic Pascal triangle $\{4, 5\}$. Moreover, in a special case the Fibonacci sequence appears, as well. This interesting result also inspired us to examine zig-zag paths on certain parts of the Euclidean square mosaic.

Consider the Euclidean square lattice and take k consecutive pieces of squares. This is the 0th layer of the k–zig-zag shape. The upper corners are the 1st, 2nd, ..., kth and $(k+1)$st vertices according to Figure 1. Extend this by an extra 0th vertex, which is the base vertex. We color it by yellow in the figures, and we join it to the 1st vertex by an extra edge. We denote the vertices of the 0th line by small boxes in Figure 1. Now move the 0the layer to reach the right-down position in the square lattice to obtain the 1st layer, and repeat this procedure with the latest layer infinitely many times. Thus, we define the square k–zig-zag shape or graph, where $k \geq 1$ is the size of the array. Finally, we label the vertices such that a label gives the number of different shortest paths from the base vertex. Figure 2 illustrates the first few layers of the square 4–zig-zag digraph, the vertices are denoted by shaded boxes with their label values and the directed edges are the black arrows. (Certain black arrows are re-colored by red for some reason. There are also particular blue arrows in the Figure; their role will be discussed

[1]Email: nemeth.laszlo@uni-sopron.hu. Research of L. N. supported by MEC_R_21 141018.
[2]Email: szalay.laszlo@uni-sopron.hu.

later.) Let $a_{i,j}$ denote the label of the vertex located in ith row and jth position ($0 \le j \le k+1$, $0 \le i$). Clearly, the fundamental rule of the construction is given by

$$a_{i,j} = \begin{cases} 1, & \text{if } i = 0; \\ a_{i-1,1}, & \text{if } j = 0, 1 \le i; \\ a_{i,j-1} + a_{i-1,j+1}, & \text{if } 1 \le j \le k, 1 \le i; \\ a_{i,k}, & \text{if } j = k+1, 1 \le i. \end{cases} \tag{1}$$

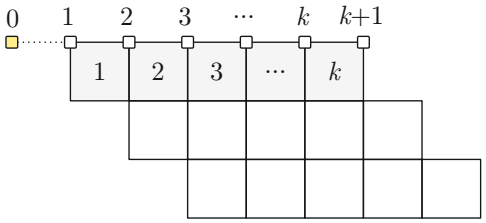


Figure 1: Zig-zag shape

For fixed $k \ge 1$ and given $0 \le j \le k+1$, let $A_j^{(k)}$ be the sequence defined by $A_j^{(k)} = (a_{i,j})_{i=0}^{\infty}$. The sequence $A_j^{(k)}$ is the jth right-down diagonal sequence of the square k–zig-zag shape. In Figure 2, the blue arrows represent the sequence $A_1^{(4)}$. We found $A_0^{(k)} = (1, A_1^{(k)})$ and $A_k^{(k)} = A_{k+1}^{(k)}$.

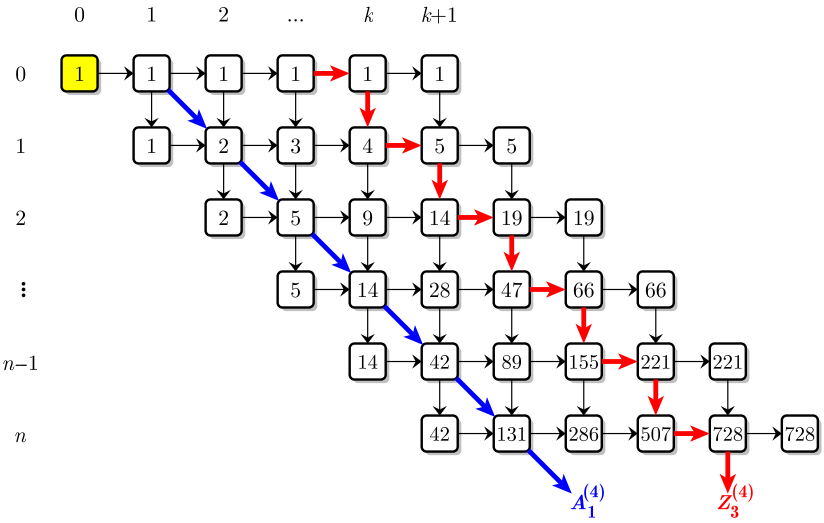


Figure 2: Square 4–zig-zag digraph ($k = 4$)

Let $Z_j^{(k)}$, $j \in \{0, 1, \ldots, k\}$ be the jth zig-zag sequence of the square k–zig-zag shape, where $Z_j^{(k)}$ is the merged sequence of $A_j^{(k)}$ and $A_{j+1}^{(k)}$. (In Figure 2, the red arrows represent the zig-zag sequence $Z_3^{(4)}$.) More precisely, $Z_j^{(k)} = (z_{i,j})_{i=0}^{\infty}$, where

$$z_{i,j} = \begin{cases} a_{\ell,j}, & \text{if } i = 2\ell; \\ a_{\ell,j+1}, & \text{if } i = 2\ell + 1. \end{cases} \tag{2}$$

Since $Z_0^{(k)}$ and $Z_k^{(k)}$ are the 'double' of $A_0^{(k)}$ and $A_k^{(k)}$, respectively, usually we examine sequences for $j \in \{1, 2, \ldots, k-1\}$.

Now we record the two main theorems of this paper. The second one is a simple corollary of the first one.

Theorem 1 (Main theorem)**.** *Given $k \geq 1$. Then all the right-down diagonal sequences $A_j^{(k)}$ for $j \in \{0, 1, \ldots, k, k+1\}$ have the same $(\left\lfloor \frac{k}{2} \right\rfloor + 1)$-th order homogeneous linear recurrence relation*

$$a_{n,j} = \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} (-1)^i \binom{k+1-i}{i+1} a_{n-1-i,j}, \qquad n \geq \left\lfloor \frac{k}{2} \right\rfloor + 1.$$

Theorem 2. *Fixing $k \geq 1$, the zig-zag sequences $Z_j^{(k)}$ for $j \in \{0, 1, \ldots, k\}$ satisfy a $(2\left\lfloor \frac{k}{2} \right\rfloor + 2)$-th order homogeneous linear recurrence relation given by*

$$z_{n,j} = \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} (-1)^i \binom{k+1-i}{i+1} z_{n-1-2i,j}, \qquad n \geq 2\left\lfloor \frac{k}{2} \right\rfloor + 2.$$

2 Recurrence relations of the square zig-zag shapes

We find that any item $a_{n,j}$, $(n \geq 1)$ is the sum of the certain items of $(n-1)$st row. More precisely, if $0 < j < k+1$, then

$$a_{n,j} = a_{n-1,j+1} + a_{n,j-1} = a_{n-1,j+1} + a_{n-1,j} + a_{n,j-2} = \cdots = \sum_{\ell=1}^{j+1} a_{n-1,\ell}. \tag{3}$$

Consider (3) for all $j \in \{1, 2, \ldots, k+1\}$ we obtain the system

$$\mathbf{v}_n = \mathbf{M} \cdot \mathbf{v}_{n-1}, \quad n \geq 1, \tag{4}$$

where

$$\mathbf{v}_n = \begin{pmatrix} a_{n,1} \\ a_{n,2} \\ a_{n,3} \\ \vdots \\ a_{n,k} \\ a_{n,k+1} \end{pmatrix}, \quad \mathbf{v}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{M}^{(k+1)\times(k+1)} = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

We know from [5] and [7] that the characteristic polynomial of any recurrence sequence (r) defined by the linear combination of the recurrence sequences $(a_j) = (a_{i,j})_{i \geq 0}$ of system (4), moreover the characteristic polynomial of the coefficient matrix $\mathbf{M}$ of system (4), coincide. Consequently, we have to determine the characteristic polynomial $p_k(x)$ of $\mathbf{M}$, and then $p_k(x)$ yields the common recurrence relation of the sequences $\{a_j\}$ and their linear combinations.

Since

$$p_k(x) = |x\mathbf{I} - \mathbf{M}|,$$

where $\mathbf{I}$ is the appropriate unit matrix, we obtain

$$\begin{aligned} p_0(x) &= x - 1, \\ p_1(x) &= \begin{vmatrix} x-1 & -1 \\ -1 & x-1 \end{vmatrix} = x^2 - 2x, \end{aligned}$$

and for $p_k(x)$ some calculation (for details see [8]) we have the binary recurrence relation

$$p_k(x) = x \cdot p_{k-1}(x) - x \cdot p_{k-2}(x), \quad k \geq 2. \tag{5}$$

Because each recurrence coefficient in (5) is one of $\pm x$, the factorization of $p_k(x)$ contains a factor x^m for some positive integer m. The next theorem provides, among others, the precise exponent m in the factorization of $p_k(x)$.

Theorem 3. *The characteristic polynomials $p_k(x)$ can be given by*

$$p_k(x) = x^{\lceil \frac{k}{2} \rceil} \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor + 1} (-1)^i \binom{k+2-i}{i} x^{\lfloor \frac{k}{2} \rfloor + 1 - i}, \qquad k \geq 0.$$

By the help of Theorem 3 we are ready to give the recurrences of the right-down diagonal sequences $A_j^{(k)}$ given Theorem 1.

2.1 Sum of rows, columns, and left-down diagonal sequences

Let $R^{(k)} = (r_n^{(k)})$ be the sum sequence of the values of the nth row of square k–zig-zag shape (see Figure 3). Considering the partial sum relation (3) we obtain

$$r_n^{(k)} = \sum_{j=0}^{k+1} a_{n,j} = a_{n,0} + a_{n+1,k}.$$

So, the recurrence sequence $r_n^{(k)}$ is the linear combination of sequences $A_n^{(k)}$, therefore they have the same characteristic polynomial and the same recurrence relation.

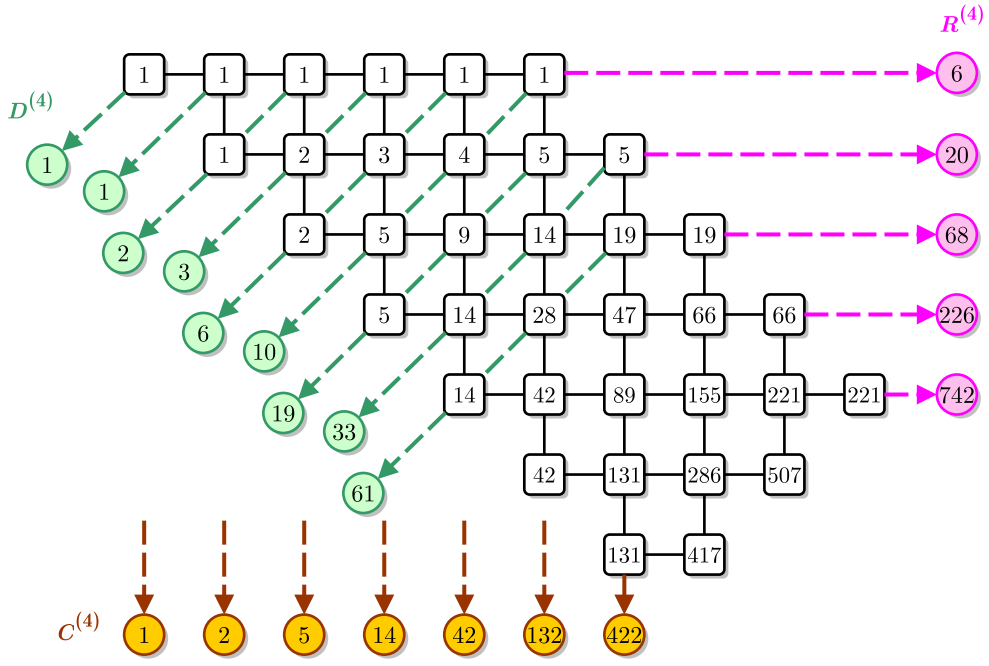


Figure 3: Sum of rows, columns, and left-down diagonal sequences

Let $C^{(k)} = (c_n^{(k)})$ be the sum sequence of columns. As $a_{n+1,0} = a_{n,1} = a_{n,0} + a_{n-1,2} = a_{n,0} + a_{n-1,1} + a_{n-2,3} = \cdots = \sum_{j=0}^{\min\{n,k+1\}-1} a_{n-j,j}$, then

$$c_n^{(k)} = \sum_{j=0}^{\min\{n,k+1\}} a_{n-j,j} = \begin{cases} a_{n+1,0}, & \text{if } n \leq k; \\ a_{n+1,0} + a_{n-k-1,k+1}, & \text{if } n > k. \end{cases}$$

Let $D^{(k)} = (d_n^{(k)})$ be the left-down diagonal sequence, where

$$d_n = \begin{cases} \sum\limits_{\ell=0}^{\ell\le\frac{n}{2},\,2\ell\le k} a_{\frac{n}{2}-\ell,2\ell}, & \text{if } n \text{ is even;} \\ \sum\limits_{\ell=0}^{\ell\le\frac{n}{2},\,2\ell\le k+1} a_{\frac{n}{2}-\ell,2\ell+1}, & \text{if } n \text{ is odd.} \end{cases}$$

Since all the $A_j^{(k)}$ sequence satisfy the same recurrence relation, then $C^{(k)}$ and $D^{(k)}$ are so.

3 Spacial zig-zag cube graphs

Now, we define a chain of cubes as an infinite part of the cube grid in the 3-dimensional space. Given a cube as the first item of the chain. Chose one of its vertices as a base vertex of our construction. (This vertex is denoted by a_0 in the righ-hand side of Figure 4.) Let the second cube be the cube having a common face with the first and having the base vertex, then let the third be the cube having a common face with the second and a common edge with the first. Let the fourth one have a common face, edge and only one vertex with the third, second and first one, respectively, and so on. That way, generally, the nth cube has exactly one common face, edge, vertex with the $(n-1)$th, $(n-2)$th, $(n-3)$th cube of the chain, respectively. Now we associate the vertices with positive integer, which gives the numbers of the shortest ways to this vertex from the base vertex of the first cube. For the first eight cubes and the values of vertices of the chain, see the left-hand-side of Figure 4.

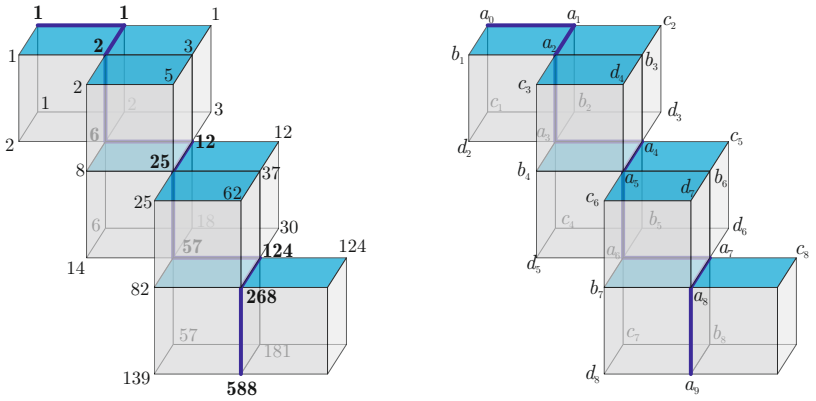


Figure 4: A cube zig-zag shape

According to the right-hand side of Figure 4 we can define zig-zag sequences a_i, b_i, c_i and d_i associated to the vertices of the cube chain. When we reconsider the vertices, edges and their sequences to a directed graph form, we gain Figure 5. Solving the system of recurrence equations, we obtain Theorem 4.

Theorem 4. *The sequence $(a_i)_{i=0}^{\infty}$ satisfies the fourth-order linear homogeneous recurrence relation*

$$a_{i+1} = a_i + a_{i-1} + 3a_{i-2} + a_{i-3}, \quad (i \ge 3)$$

with initial values $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 6$.

4 Conclusions

With a special directed zig-zag graph defined on a part of the square grid we generally determined the recurrences of some special directional sequences associated to the graph. We give new combinatorial

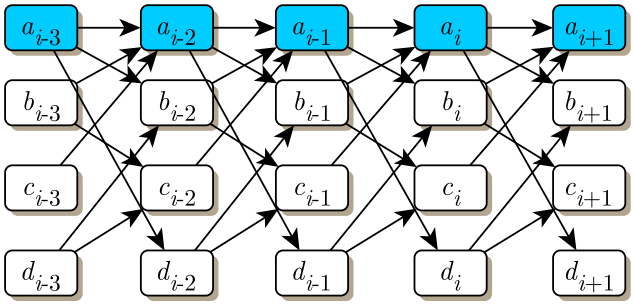


Figure 5: Digraph form of the system of recurrences

interpretations to more then forty sequences appearing in OEIS [9]. For example, if $k = 4$, then $A_1^{(4)} =$ A080937 $A_2^{(4)} =$ A094790, $A_3^{(4)} =$ A094789, $A_4^{(4)} =$ A005021, $Z_2^{(4)} =$ A006053, $(d_{2n}^{(4)}) =$ A052975, $(d_{2n+1}^{(4)}) =$ A060557, $D^{(4)} =$ A028495, and in case of cube zig-zag sequences: A214663, A232162, A232164, A232165.

References

[1] S. Ahmad, H. M. A. Siddiqui, A. Ali, M. R. Farahani, M. Imran, and I. N. Cangul. "On Wiener index and Wiener polarity index of some polyomino chains". In: *J. Discrete Math. Sci. Cryptogr.* 22.7 (2019), pp. 1151–1164. DOI: 10.1080/09720529.2019.1688965.

[2] Y. Baryshnikov and D. Romik. "Enumeration formulas for Young tableaux in a diagonal strip". In: *Israel J. Math.* 178 (2010), pp. 157–186. DOI: 10.1007/s11856-010-0061-6.

[3] H. Belbachir, L. Németh, and L. Szalay. "Hyperbolic Pascal triangles". In: *Appl. Math. Comput.* 273 (2016), pp. 453–464. DOI: 10.1016/j.amc.2015.10.001.

[4] H. Belbachir and L. Szalay. "On the arithmetic triangles". In: *Šiauliai Math. Semin.* 9(17) (2014), pp. 15–26.

[5] L. Németh. "The growing ratios of hyperbolic regular mosaics with bounded cells". In: *Armen. J. Math.* 9.1 (2017), pp. 1–19.

[6] L. Németh and L. Szalay. "Recurrence sequences in the hyperbolic Pascal triangle corresponding to the regular mosaic $\{4, 5\}$". In: *Ann. Math. Inform.* 46 (2016), pp. 165–173.

[7] L. Németh and L. Szalay. "Power sums in hyperbolic Pascal triangles". In: *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat.* 26.1 (2018), pp. 189–203. DOI: 10.2478/auom-2018-0012.

[8] L. Németh and L. Szalay. "Sequences involving square zig-zag shapes". In: *J. Integer Seq.* 24.5 (2021), Art. 21.5.2, 14.

[9] OEIS Foundation Inc. *The On-Line Encyclopedia of Integer Sequences*. Published electronically at https://oeis.org. 2021.

[10] R. P. Stanley. "A survey of alternating permutations". In: *Combinatorics and graphs*. Vol. 531. Contemp. Math. Amer. Math. Soc., Providence, RI, 2010, pp. 165–196. DOI: 10.1090/conm/531/10466.

Junio, 2022

The Discrete Mathematics Days (DMD20/22) will be held on July 4-6, 2022, at Facultad de Ciencias of the Universidad de Cantabria (Santander, Spain). The main focus of this international conference is on current topics in Discrete Mathematics, including (but not limited to):

Algorithms and Complexity
Combinatorics
Coding Theory
Cryptography
Discrete and Computational Geometry
Discrete Optimization
Graph Theory
Location and Related Problems

The previous editions were held in Sevilla in 2018 and in Barcelona in 2016, inheriting the tradition of the Jornadas de Matemática Discreta y Algorítmica (JMDA), the Spanish biennial meeting (since 1998) on Discrete Mathematics. The program consists on four plenary talks, 42 contributed talks and a poster session with 11 contributions.

AUTHORS

A. González, A. Mucherino, Abel Cabrera-Martínez, Adriana Dapena, Aida Abiad, Albert Atserias, Alberto Espuny Díaz, Alberto Márquez, Alexander Allin, Amanda Montejano, Ana Isabel Gómez Pérez, Ander Lamaison, Andrew Goodall, Anna de Mier, Anna Slivková, Antonina P. Khramova, Antonio Cañete, Antonio Macchia, Anuj Dawar, Anusch Taraz, Aranka Hrušková, Arnau Padrol, Béla Bajnok, Benedetto Manca, Bernardo González Merino, Bojan Bašić, Boris Brimkov, Bruno Benedetti, C. Hernando, Carlos Marijuán, Christian Haase, Christine Bachoc, Christoph Spiegel, Clément Requilé, Cristina Dalfó, Daniel Král, Danny Vagnozzi, Dante Luber, David Fabian, David Orden, Davide Bolognini, Delia Garijo, Deniz Ağaoğlu Çağırıcı, Domingo Gómez Pérez, Edgardo Roldán-Pensado, Eduardo Lucas Marín, Eugen Mandrescu, Eva Philippe, Fábio Botler, Francesco Strazzanti, Francisco Santos Leal, Francisco Vázquez-Araujo, Francisco-Javier Soto, Gabriela Araujo-Pardo, Georg Loho, Germain Poullot, Giancarlo Rinaldo, Gilad Chase, Giulia Codenotti, Guillem Perarnau, Haydee Jiménez-Tafur, Hong Liu, Irene Gil Fernández, Isabel Fernández, J. David Suárez, Jack H. Koolen, Jacob Lahne, János Pach, Johannes Hofscheier, Jordi Castellví, Jorge Olarte, José Aliste-Prieto, José Manuel Jiménez-Cobano, José María Ucha-Enríquez, José Zamora, Josep Conde, Josep Díaz, Joseph Hyde, Josué Tonelli-Cueto, Jozefien D'haeseleer, Juan Alberto Rodríguez-Velázquez, Juanjo Rué, Julia Wolf, Julian Pfeifle, Katherine Phetxumphou, L. Boza, László Németh, László Szalay, Leo Liberti, Lidija Čomić, Lluis Vena Cros, Luis Crespo Ruiz, Luis Felipe Tabera Alonso, M. I. Sanz, M. Mora, M. P. Revuelta, Magda Dettlaff, Magdalena Lemańska, Manuel A. Espinosa-García, Manuel Radons, Marc Noy, María José Souto-Salorio, Maria Serna, Marie-Charlotte Brandenburg, Marino Tejedor-Romero, Mario Huicochea, Marta Pavelka, Marthe Bonamy, Márton Borbényi, Maximilian Wötzel, Maya Stein, Michael Joswig, Miguel Àngel Fiol, Miquel Ortega, Miriam Pisonero, Nacho López, Nathan Lindzey, Neta Dafni, Ngoc Mai Tran, Öznur Yaşar Diner, Olaf Parczyk, Onur Çağırıcı, Oriol Serra, Panna Fekete, Paola Magillo, Pascal Schweitzer, Patrick Morris, Péter Pál Pach, Phablo F. S. Moura, Philippe Nadeau, Pierre-Louis Poirion, Robin Simoens, Rosa Orellana, Sakander Hayat, Sean Prendiville, Sebastian Pokutta, Sergio Cabello, Shalom Eliahou, Simon B. Hengeveld, Stephan Gardoll, T. Ahmed, Tássio Naia, Thomas Hall, Thorsten Theobald, Tibor Szabó, Vadim E. Levit, Vincent Pilaud, Yuval Filmus.

UNIVERSIDAD DE CANTABRIA
FACULTAD DE CIENCIAS
CANTABRIA CAMPUS INTERNACIONAL
MINISTERIO DE CIENCIA E INNOVACIÓN
AGENCIA ESTATAL DE INVESTIGACIÓN

ISBN 978-84-19024-03-9
0 €
www.editorial.unican.es
THEMA: PBD, 1DSE-ES-F, 3MRBH